NUMBER THEORY
FALL 2022

ASSIGNMENT 1.1
DUE FEBRUARY 7

Exercise 1. Let $a, b \in \mathbb{Z}$. Recall the "useful fact" that if $|b| > 1$ and $a \neq 0$, then $|a| < |ab|$. Prove that the converse implication is also true.

Exercise 2. Give a careful proof that the only elements of $\mathbb{Z}$ with multiplicative inverses are the members of the set

$$\mathbb{Z}^\times := \{\pm 1\}.$$

Exercise 3. Given pairs of natural numbers $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$, define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by

$$(a, b) \sim (c, d) \quad \Leftrightarrow \quad a + d = b + c.$$

a. Show that $\sim$ is an equivalence relation using only the operation $+$.

b. Let $\overline{(a, b)}$ denote the equivalence class of (a, b) under $\sim$, and let $(\mathbb{N} \times \mathbb{N})/\sim$ be the set of all equivalence classes. Show that the rule $\overline{(a, b)} \mapsto a - b$ gives a well-defined bijection between $(\mathbb{N} \times \mathbb{N})/\sim$ and $\mathbb{Z}$.[1]

c. What happens if we replace $+$ in the definition of $\sim$ by $\times$?

In Zermelo-Fraenkel set theory, which is one way to axiomatize modern mathematics, every object is a set, and the axioms give rules for how these sets behave. Starting with these axioms alone, one defines $\mathbb{N}$ to be the set of cardinalities of nonempty finite sets. So where does $\mathbb{Z}$ comes from? The set $(\mathbb{N} \times \mathbb{N})/\sim$ in Exercise 3 is actually the *definition* of $\mathbb{Z}$ as a set, so that part **b** is completely circular. Nonetheless, I think it demonstrates the intuition behind the decision to declare that $(\mathbb{N} \times \mathbb{N})/\sim$ is the set of integers. It isn't difficult to also define addition and multiplication in $(\mathbb{N} \times \mathbb{N})/\sim$ using only the operation $+$ in $\mathbb{N}$, but checking that everything is well-defined is a bit tedious. The end result is the familiar ring $(\mathbb{Z}, +, \times)$, built formally from first principles.

[1]If you aren't sure what "well-defined" means, please ask. It's a very important technical term.