Exercise 1. Let $n \in \mathbb{N}$ and let $a, b \in \mathbb{Z}$. Prove that a and b leave the same remainder when divided by n if and only if $n|a - b$.

Exercise 2. Given $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{N}$, define $(a, b) \sim (c, d)$ if and only if $ad = bc$. One can show (and you should check) that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{N}$. An equivalence class under $\sim$ is called a *fraction*, and we let $\frac{a}{b}$ denote the fraction containing (a, b). Then, by definition,

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, (a, b) \in \mathbb{Z} \times \mathbb{N} \right\}$$

is the set of all fractions (is this starting to sound familiar?). Show that the map $\mathbb{Z} \to \mathbb{Q}$ defined by $n \mapsto \frac{n}{1}$ is injective.

Strictly speaking, an integer *is not* a rational number. However, Exercise 3 shows that there is a bijection between $\mathbb{Z}$ and the set

$$\mathbb{Z}_{\mathbb{Q}} = \left\{ \frac{n}{1} \,\middle|\, n \in \mathbb{Z} \right\}.$$

We will call a fraction in $\mathbb{Z}_{\mathbb{Q}}$ a *rational integer*.

Exercise 3. Let $a \in \mathbb{Z}$ and let $b \in \mathbb{N}$.

- **a.** Identifying a and b with their images in $\mathbb{Z}_{\mathbb{Q}}$, show that the equation $a = b \cdot \frac{a}{b}$ holds.
- **b.** Show that $b|a$ if and only if $\frac{a}{b} \in \mathbb{Z}_{\mathbb{Q}}$.
- **c.** Show that when $b|a$, $\frac{a}{b}$ is (the rational integer corresponding to) the divisor of a complementary to b. This is handy, since it provides (meaningful) notation for the complementary divisor that doesn't introduce a third variable.

Exercise 4. For any real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, which is the unique integer satisfying the inequality

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

Let $a \in \mathbb{Z}$ and let $b \in \mathbb{N}$, and use the division algorithm to write $a = bq + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < b$. Prove that $q = \lfloor \frac{a}{b} \rfloor$ and derive a similar expression for r. Explain how you can use this to compute q and r using a (very simple) hand held calculator.

Exercise 5. Let $a, b \in \mathbb{N}$. Prove that the natural numbers $\frac{a}{(a,b)}$ and $\frac{b}{(a,b)}$ are relatively prime.