Exercise 1. Without appealing to the existence of irreducible/prime factorizations in $\mathbb{N}$, use strong induction to prove that every $a > 1$ is divisible by an irreducible/prime. This weaker result is sufficient to prove the infinitude of the set of primes using Euclid's familiar argument.

Exercise 2. Let $k \in \mathbb{N}$ and $a > 1$. Let $a = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ be the canonical prime factorization of a. Prove that a is a kth power in $\mathbb{N}$ if and only if $k | e_j$ for all j.

Exercise 3. Given $a, b > 1$, explain why it is always possible to assume that the canonical prime factorizations of a and b involve exactly the same primes, provided we allow some of the exponents to be 0.

Exercise 4. Given $a, b > 1$, let

$$a = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$

$$b = p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r},$$

be the canonical prime factorizations of a and b, as in the preceding exercise. Prove that

$$(a, b) = p_1^{\min\{e_1, f_1\}} p_2^{\min\{e_2, f_2\}} \cdots p_r^{\min\{e_r, f_r\}}.$$

Exercise 5. Choose an enumeration $p_1, p_2, p_3, \ldots$ of the primes. Given $a \in \mathbb{N}$ write

$$a = \prod_{i=1}^{\infty} p_i^{e_i(a)}$$

for the canonical prime factorization of a, where $e_i(a) \neq 0$ for only finitely many i. Prove that the "unzipping" function $u : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by

$$a \mapsto \left(\prod_{i=1}^{\infty} p_i^{e_{2i}(a)}, \prod_{i=1}^{\infty} p_i^{e_{2i-1}(a)} \right)$$

is a bijection.