Number Theory
Spring 2022

Assignment 5.1
Due March 23

Exercise 1. Let $n \in \mathbb{N}$ and write $n = ab^2$ with $a, b \in \mathbb{N}$ and a squarefree. Prove that a and b are unique. That is, show that if $a, b, c, d \in \mathbb{N}$, a and c are squarefee, and $ab^2 = cd^2$, then $a = c$ and $b = d$.

Exercise 2. Let $k \in \mathbb{N}$, $k \geq 2$. We say that $n \in \mathbb{N}$ is *kth power free* if for every $d \in \mathbb{N}$, $d^k | n$ implies $d = 1$. Prove that every $n \in \mathbb{N}$ can be expressed uniquely in the form $n = ab^k$, where $a, b \in \mathbb{N}$ and a is kth power free.

Exercise 3. Let $k \geq 2$ be an integer. Prove that $n \in \mathbb{N}$ is kth power free if and only if $\nu_p(n) < k$ for all primes p.

Exercise 4. If $\mu : \mathbb{N} \to \{0, \pm 1\}$ denotes the Möbius function, prove that for every $n \in \mathbb{N}$ one has

$$\sum_{d^k | n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is kth power free,} \\ 0 & \text{otherwise.} \end{cases}$$

Here the sum is indexed by all $d \in \mathbb{N}$ for which $d^k | n$.