Euler Equations

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An *Euler equation* is a homogeneous second order linear ODE of the form

$$x^2y'' + axy' + by = 0, \quad x > 0, \tag{1}$$

where a and b are (real) constants. Because the coefficients on y'' and y' (x^2 and ax, respectively) are *not* constants, an Euler equation *cannot* be directly solved using the techniques for solving constant coefficient equations. However, the general solution to an Euler equation can easily be obtained by making the change of (independent) variables $x = e^t$ (or $t = \ln x$) and then *reducing* it to a constant coefficient equation.

Since the prime notation y' denotes the derivative of y with respect to x, we use $\dot{y}$ to denote differentiation with respect to t. The chain rule then gives

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \dot{y}x^{-1},\\
y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\dot{y}x^{-1})\\
&= \frac{d\dot{y}}{dx}x^{-1} - \dot{y}x^{-2} = \frac{d\dot{y}}{dt}\frac{dt}{dx}x^{-1} - \dot{y}x^{-2},\\
&= \ddot{y}x^{-2} - \dot{y}x^{-2} = (\ddot{y} - \dot{y})x^{-2}.
\end{aligned}$$

Substituting these expressions for y' and y'' into the Euler equation (1) yields

$$(\ddot{y} - \dot{y}) + a\dot{y} + by = 0 \quad \Leftrightarrow \quad \ddot{y} + (a-1)\dot{y} + by = 0, \tag{2}$$

which is a second order homogenous linear ODE for y (in the independent variable t) *with constant coefficients*! The characteristic equation for $\ddot{y} + (a-1)\dot{y} + by = 0$ is called the *indicial equation* of (1), and is traditionally written

$$\rho^2 + (a-1)\rho + b = 0. \tag{3}$$

The roots of the indicial equation are called the *indices* of (1). Historically the term *index* was used as a synonym for *exponent*, which is precisely how the roots of the indicial equation (3) are used to solve (1).

If (3) has two real roots $\rho_1 \neq \rho_2$, then the general solution to (2) is

$$y = c_1e^{\rho_1 t} + c_2e^{\rho_2 t}.$$

Since $t = \ln x$, this means the general solution to (1) in this case is

$$y = c_1x^{\rho_1} + c_2x^{\rho_2}.$$

When (3) has a single repeated real root ρ_1, the general solution to (2) is then

$$y = (c_1 t + c_2)e^{\rho_1 t}.$$

Substituting $t = \ln x$ this becomes the general solution

$$y = (c_1 \ln x + c_2)x^{\rho_1}.$$

to (1).

Finally, if (3) has nonreal complex roots $\rho = \alpha \pm i\beta$ ($\beta \neq 0$), then the general solution to (2) is

$$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

When we set $t = \ln x$ this becomes the somewhat more complicated expression

$$y = x^{\alpha}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)),$$

which is the general solution to (1) in this case. We summarize our findings below.

Theorem 1. *The general solution of the Euler equation*

$$x^2 y'' + axy' + by = 0, \quad x > 0, \tag{4}$$

is determined by the roots of its indicial equation

$$\rho^2 + (a-1)\rho + b = 0 \tag{5}$$

as follows.

a. *If the indicial equation* (5) *has two real roots* $\rho = \rho_1, \rho_2$, *then the general solution of the Euler equation* (4) *is given by*

$$y = c_1 x^{\rho_1} + c_2 x^{\rho_2},$$

where c_1 *and* c_2 *are arbitrary (real) constants.*

b. *If the indicial equation* (5) *has a single (repeated) real root* $\rho = \rho_1$, *then the general solution of the Euler equation* (4) *is given by*

$$y = (c_1 \ln x + c_2)x^{\rho_1},$$

where c_1 *and* c_2 *are arbitrary (real) constants.*

c. *If the indicial equation has (nonreal) complex roots* $\rho = \alpha \pm i\beta$ ($\beta \neq 0$), *then the general solution of the Euler equation* (4) *is given by*

$$y = x^{\alpha}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)),$$

where c_1 *and* c_2 *are arbitrary (real) constants.*

Example. Solve the Euler equation

$$x^2 y'' - 2xy' + 2y = 0, \quad x > 0.$$

Solution. The indicial equation is

$$\rho^2 + (-2 - 1)\rho + 2 = \rho^2 - 3\rho + 2 = 0.$$

Because

$$\rho^2 - 3\rho + 2 = (\rho - 1)(\rho - 2),$$

the indicial equation has the distinct real roots $\rho = 1, 2$. According to Theorem 1, this means the general solution of the ODE is

$$\boxed{y = c_1 x + c_2 x^2.}$$

□

Example. Solve the Euler equation

$$x^2 y'' + 5xy' + 4y = 0, \quad x > 0.$$

Solution. The indicial equation is

$$\rho^2 + (5 - 1)\rho + 4 = \rho^2 + 4\rho + 4 = 0.$$

Because

$$\rho^2 + 4\rho + 4 = (\rho + 2)^2,$$

the indicial equation has the single repeated roots $\rho = -2$. According to Theorem 1, this means the general solution of the ODE is

$$\boxed{y = (c_1 \ln x + c_2)x^{-2}.}$$

□

Example. Solve the Euler equation

$$x^2 y'' + xy' + 3y = 0, \quad x > 0.$$

Solution. The indicial equation is

$$\rho^2 + (1 - 1)\rho + 3 = \rho^2 + 3 = 0,$$

whose roots are $\rho = \pm i\sqrt{3} = 0 \pm i\sqrt{3}$. According to Theorem 1, this means the general solution of the ODE is

$$\boxed{y = c_1 \cos(\sqrt{3} \ln x) + c_2 \sin(\sqrt{3} \ln x).}$$

The factor x^α of Theorem 1 isn't present since in this case $\alpha = 0$ and $x^0 = 1$ for $x > 0$. □