Exercise 1. If u is a function of x and t with continuous second order partial derivatives, and we set

$$\alpha = ax + bt, \quad \beta = mx + nt$$

with $an - bm \neq 0$, use the chain rule to show that

$$u_{tt} = b^2 u_{\alpha\alpha} + 2bn u_{\alpha\beta} + n^2 u_{\beta\beta}.$$

[*Suggestion.* Apply the chain rule once to compute u_t in terms of u_α and u_β, as we did in class. Then apply the chain rule again, in the exact same way, to compute $(u_\alpha)_t$ and $(u_\beta)_t$.]

Exercise 2. Solve the 1-D wave equation (for an infinitely long string) subject to the given initial data.

a. $u(x,0) = f(x)$, $u_t(x,0) = 0$

b. $u(x,0) = \dfrac{1}{1+x^2}$, $u_t(x,0) = -2xe^{-x^2}$

c. $u(x,0) = e^{-x^2}$, $u_t(x,0) = \dfrac{x}{(1+x^2)^2}$

Exercise 3. If λ is a constant, verify that $u(x,t) = \sin(\lambda x)\cos(\lambda ct)$ solves the 1-D wave equation $u_{tt} = c^2 u_{xx}$. For which values of λ does this function also satisfy the boundary conditions $u(0,t) = u(L,t) = 0$?

Exercise 4. Suppose that $u(x,t)$ and $v(x,t)$ have continuous second order partial derivatives and are related through the equations

$$\frac{\partial u}{\partial t} = -A\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial t} = -B\frac{\partial u}{\partial x}$$

for some positive constants A and B. Show that u and v are both solutions of the 1-D wave equation with $c = \sqrt{AB}$.