Consider the heat boundary value problem

$$\begin{aligned} &u_t = c^2 u_{xx}, && t > 0,\ \ 0 < x < L, && (1)\\ &u_x(0,t) = \kappa_1 u(0,t),\ \ u_x(L,t) = -\kappa_2 u(L,t), && t > 0, && (2)\\ &u(x,0) = f(x), && 0 \le x \le L, && (3) \end{aligned}$$

in which κ_1 and κ_2 are positive constants. Solutions to this problem describe the flow of thermal energy in a thin conductive rod which is radiating energy at both of its endpoints.

Exercise 1. Use separation of variables to show that any solution to (1) and (2) of the form $u(x,t) = X(x)T(t)$ must satisfy

$$T' - kc^2 T = 0,$$
$$X'' - kX = 0,\ \ X'(0) = \kappa_1 X(0),\ \ X'(L) = -\kappa_2 X(L),$$

where k is the separation constant.

Exercise 2. Show that if X is nontrivial, then $k = -\mu^2 < 0$ and $\mu > 0$ must solve the equation

$$\tan(\mu L) = \frac{(\kappa_1 + \kappa_2)\mu}{\mu^2 - \kappa_1 \kappa_2}.$$

Exercise 3. Show that the corresponding solutions are

$$X_n(x) = \mu_n \cos(\mu_n x) + \kappa_1 \sin(\mu_n x)$$

and

$$T_n(t) = e^{-c^2 \mu_n^2 t}$$

for $n \in \mathbb{N}$, where μ_n is the nth positive solution to the equation

$$\tan(\mu L) = \frac{(\kappa_1 + \kappa_2)\mu}{\mu^2 - \kappa_1 \kappa_2}. \tag{1}$$

Exercise 4. Use a graph to show that equation (1) has infinitely many positive solutions. Approximate $\mu_1, \mu_2, \mu_3, \mu_4$ and μ_5 in the case that $L = \pi$ and $\kappa_1 = \kappa_2 = 1$.

Exercise 5. Show that the functions $X_1, X_2, X_3, \ldots$ are pairwise orthogonal on the interval $[0, L]$.

Exercise 6. Conclude that the complete solution to the heat problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x), \tag{2}$$

where

$$c_n = \frac{1}{\tau_n} \int_0^L f(x) X_n(x)\, dx \quad \text{and} \quad \tau_n = \int_0^L X_n^2(x)\, dx.$$

Exercise 7. Using a partial sum with 100 terms, animate the solution (2) in time, taking $c = 1$, $L = \pi$, $\kappa_1 = 2$, $\kappa_2 = 1$, and $f(x) = x^2(\pi - x)$. Use the time interval $0 \leq t \leq 10$ with time steps of size $\Delta t = 0.1$.

Instructions

- You may work in a group of up to 3 students. Each student in a group is expected to contribute equally to the project. If you feel that one of your group members is not doing his or her share, please let me know.
- Written solutions to the exercises must be typed into a single document, and exported in PDF format. The plot for Exercise 4 should be included in the body of this document.
- The animation for Exercise 7 may be coded using any language/program you prefer (I've been using Maple in class). The animation itself must be submitted as an independent file using a standard graphics file format (GIF, for instance). The code itself must be submitted as a separate text file.
- Each group must upload a single Zip file (containing the 3 files described above) to `http://tlearn.trinity.edu` no later than midnight on March 22. The name of each group's Zip file should include the last names of every member of that group. Late projects *will not be accepted.*
- Failure to adhere to these guidelines will be penalized. If you have any questions or concerns, please ask me.