The two-dimensional heat equation

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Partial Differential Equations
Lecture 12

Physical motivation

Goal: Model heat flow in a two-dimensional object (thin plate).

Set up: Represent the plate by a region in the *xy*-plane and let

$$u(x,y,t) = \begin{cases} \text{temperature of plate at position } (x,y) \text{ and} \\ \text{time } t. \end{cases}$$

For a fixed t, the height of the surface $z = u(x,y,t)$ gives the temperature of the plate at time t and position (x,y).

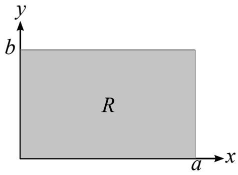
Under ideal assumptions (e.g. uniform density, uniform specific heat, perfect insulation along faces, no internal heat sources etc.) one can show that u satisfies the *two dimensional heat equation*

$$u_t = c^2 \Delta u = c^2(u_{xx} + u_{yy})$$

Rectangular plates and boundary conditions

For now we assume:

- The plate is rectangular, represented by $R = [0, a] \times [0, b]$.



- The plate is imparted with some initial temperature:

$$u(x, y, 0) = f(x, y), \quad (x, y) \in R.$$

- The edges of the plate are held at zero degrees:

$$\begin{aligned} u(0, y, t) = u(a, y, t) = 0, &\qquad 0 \le y \le b,\ t > 0, \\ u(x, 0, t) = u(x, b, t) = 0, &\qquad 0 \le x \le a,\ t > 0. \end{aligned}$$

Separation of variables

We seek nontrivial solutions of the form

$$u(x, y, t) = X(x)Y(y)T(t).$$

Plugging this into $u_t = c^2(u_{xx} + u_{yy})$ we get

$$XYT' = c^2\left(X''YT + XY''T\right) \quad \Rightarrow \quad \frac{T'}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

Because the two sides are functions of different independent variables, they must be constant:

$$\frac{T'}{c^2T} = A = \frac{X''}{X} + \frac{Y''}{Y} \quad \Rightarrow \quad \begin{cases} T' - c^2AT = 0, \\ \dfrac{X''}{X} = -\dfrac{Y''}{Y} + A. \end{cases}$$

Since the two sides again involve unrelated variables, both are constant:

$$\frac{X''}{X} = B = -\frac{Y''}{Y} + A.$$

Setting $C = A - B$, these equations can be rewritten as

$$X'' - BX = 0, \quad Y'' - CY = 0.$$

The first boundary condition is

$$0 = u(0, y, t) = X(0)Y(y)T(t).$$

Canceling Y and T yields $X(0) = 0$. Likewise, we obtain

$$X(a) = 0, \quad Y(0) = Y(b) = 0.$$

There are no boundary conditions on T.

We have already solved the two boundary value problems for X and Y. The nontrivial solutions are

$$X = X_m(x) = \sin(\mu_m x), \qquad \mu_m = \frac{m\pi}{a}, \qquad m \in \mathbb{N},$$
$$Y = Y_n(y) = \sin(\nu_n y), \qquad \nu_n = \frac{n\pi}{b}, \qquad n \in \mathbb{N},$$

with separation constants $B = -\mu_m^2$ and $C = -\nu_n^2$.

Since $T' - c^2 AT = 0$, and $A = B + C = -\left(\mu_m^2 + \nu_n^2\right) < 0$,

$$T = T_{mn}(t) = T = T_{mn}(t) = A_{mn} e^{-\lambda_{mn}^2 t},$$

where

$$\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Superposition

Assembling these results, we find that for any *pair* $m, n \geq 1$ we have the normal mode

$$u_{mn}(x, y, t) = X_m(x) Y_n(y) T_{mn}(t) = A_{mn} \sin(\mu_m x) \sin(\nu_n y) e^{-\lambda_{mn}^2 t}.$$

The principle of superposition gives the general solution

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(\mu_m x) \sin(\nu_n y) e^{-\lambda_{mn}^2 t}.$$

The initial condition requires that

$$f(x, y) = u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right),$$

which is a so-called *double Fourier series* for $f(x, y)$.

Representability

Which functions are given by double Fourier series?

The following result partially answers this first question.

Theorem

If $f(x,y)$ is a C^2 function on the rectangle $[0,a]\times[0,b]$, then

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

for appropriate B_{mn}.

- To say that $f(x,y)$ is a C^2 *function* means that f as well as its first and second order partial derivatives are all continuous.
- While not as general as the Fourier representation theorem, this result is sufficient for our applications.

Orthogonality (again!)

How can we compute the coefficients in a double Fourier series?

The following result helps us answer this second question.

Theorem

The functions

$$Z_{mn}(x,y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \quad m,n \in \mathbb{N}$$

are pairwise orthogonal relative to the inner product

$$\langle f, g\rangle = \int_0^a \int_0^b f(x,y)g(x,y)\,dy\,dx.$$

This is easily verified using the orthogonality of the functions $\sin(n\pi x/p)$ on the interval $[0,p]$.

Using the usual argument, it follows that if

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \underbrace{\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}_{Z_{mn}},$$

then

$$B_{mn} = \frac{\langle f, Z_{mn}\rangle}{\langle Z_{mn}, Z_{mn}\rangle} = \frac{\int_0^a \int_0^b f(x,y) Z_{mn}(x,y)\, dy\, dx}{\int_0^a \int_0^b Z_{mn}(x,y)^2\, dy\, dx}$$

$$= \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)\, dy\, dx.$$

So, we can finally write down the complete solution to our original problem.

Conclusion

Theorem

If $f(x,y)$ is a "sufficiently nice" function on $[0,a]\times[0,b]$, then the solution to the heat equation with homogeneous Dirichlet boundary conditions and initial condition $f(x,y)$ is

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin(\mu_m x)\,\sin(\nu_n y)\,e^{-\lambda_{mn}^2 t},$$

where $\mu_m=\dfrac{m\pi}{a}$, $\nu_n=\dfrac{n\pi}{b}$, $\lambda_{mn}=c\sqrt{\mu_m^2+\nu_n^2}$, *and*

$$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\mu_m x)\sin(\nu_n y)\,dy\,dx.$$

Example

A 2×2 square plate with $c = 1/3$ is heated in such a way that the temperature in the lower half is 50, while the temperature in the upper half is 0. After that, it is insulated laterally, and the temperature at its edges is held at 0. Find an expression that gives the temperature in the plate for $t > 0$.

We must solve the heat problem above with $a = b = 2$ and

$$f(x,y) = \begin{cases} 50 & \text{if } y \leq 1, \\ 0 & \text{if } y > 1. \end{cases}$$

The coefficients in the solution are

$$\begin{aligned} A_{mn} &= \frac{4}{2 \cdot 2} \int_0^2 \int_0^2 f(x,y) \sin\left(\frac{m\pi}{2}x\right) \sin\left(\frac{n\pi}{2}y\right)\, dy\, dx \\ &= 50 \int_0^2 \sin\left(\frac{m\pi}{2}x\right)\, dx \int_0^1 \sin\left(\frac{n\pi}{2}y\right)\, dy \end{aligned}$$

$$= 50 \left(\frac{2(1 + (-1)^{m+1})}{\pi m} \right) \left(\frac{2(1 - \cos \frac{n\pi}{2})}{\pi n} \right)$$
$$= \frac{200}{\pi^2} \frac{(1 + (-1)^{m+1})(1 - \cos \frac{n\pi}{2})}{mn}.$$

Since $\lambda_{mn} = \frac{\pi}{3}\sqrt{\frac{m^2}{4} + \frac{n^2}{4}} = \frac{\pi}{6}\sqrt{m^2 + n^2}$, the solution is

$$u(x, y, t) = \frac{200}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{(1 + (-1)^{m+1})(1 - \cos \frac{n\pi}{2})}{mn} \sin\left(\frac{m\pi}{2}x\right) \times \sin\left(\frac{n\pi}{2}y\right) e^{-\pi^2(m^2+n^2)t/36} \right).$$

Inhomogeneous boundary conditions

Steady state solutions and Laplace's equation

2-D heat problems with inhomogeneous Dirichlet boundary conditions can be solved by the "homogenizing" procedure used in the 1-D case:

1. Find and subtract the steady state ($u_t \equiv 0$);
2. Solve the resulting homogeneous problem;
3. Add the steady state to the result of Step 2.

We will focus only on finding the steady state part of the solution. Setting $u_t = 0$ in the 2-D heat equation gives

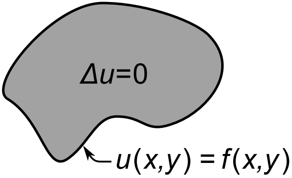
$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{(Laplace's equation)},$$

solutions of which are called *harmonic functions.*

Dirichlet problems

Definition: The *Dirichlet problem* on a region $R \subseteq \mathbb{R}^2$ is the boundary value problem

$$\Delta u = 0 \quad \text{inside } R,$$
$$u(x,y) = f(x,y) \quad \text{on } \partial R.$$



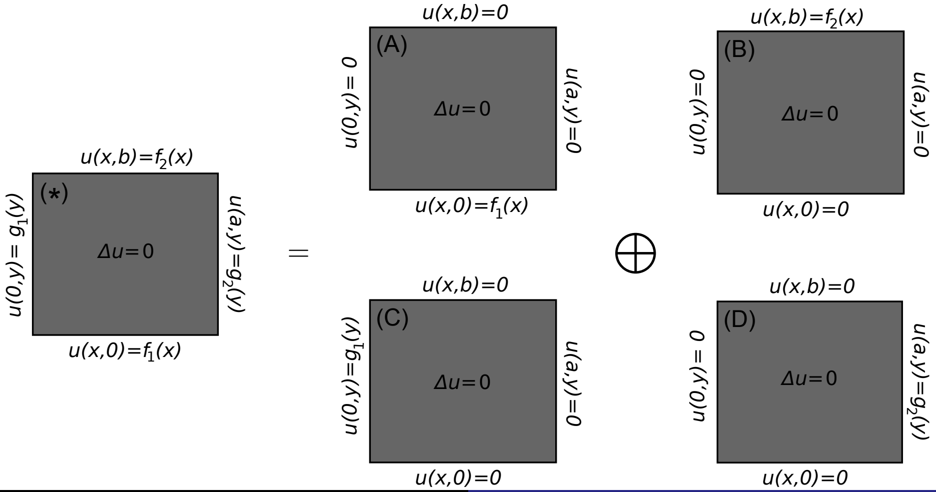
When the region is a rectangle $R = [0,a] \times [0,b]$, the boundary conditions will be given on each edge separately as:

$$u(x,0) = f_1(x), \qquad u(x,b) = f_2(x), \qquad 0 < x < a,$$
$$u(0,y) = g_1(y), \qquad u(a,y) = g_2(y), \qquad 0 < y < b.$$

Solving the Dirichlet problem on a rectangle

'Homogenization and superposition

Strategy: Reduce to four simpler problems and use superposition.



Remarks:

- If u_A, u_B, u_C and u_D solve the Dirichlet problems (A), (B), (C) and (D), then the solution to $(*)$ is

$$u = u_A + u_B + u_C + u_D.$$

- Note that the boundary conditions in (A) - (D) are all homogeneous, with the exception of a single edge.

- Problems with inhomogeneous Neumann or Robin boundary conditions (or combinations thereof) can be reduced in a similar manner.

Solution of the Dirichlet problem on a rectangle

Case B

Goal: Solve the boundary value problem (B):

$$\begin{aligned} \Delta u &= 0, && 0 < x < a,\ 0 < y < b, \\ u(x,0) &= 0,\ u(x,b) = f_2(x), && 0 < x < a, \\ u(0,y) &= u(a,y) = 0, && 0 < y < b. \end{aligned}$$

Setting $u(x,y) = X(x)Y(y)$ leads to

$$\begin{aligned} X'' + kX &= 0, & Y'' - kY &= 0, \\ X(0) = X(a) &= 0, & Y(0) &= 0. \end{aligned}$$

We know the nontrivial solutions for X are given by

$$X(x) = X_n(x) = \sin(\mu_n x), \quad \mu_n = \frac{n\pi}{a}, \quad k = \mu_n^2 \quad (n \in \mathbb{N}).$$

Interlude

The hyperbolic trigonometric functions

The *hyperbolic cosine and sine functions* are

$$\cosh y = \frac{e^y + e^{-y}}{2}, \quad \sinh y = \frac{e^y - e^{-y}}{2}.$$

They satisfy the following identities:

$$\cosh^2 y - \sinh^2 y = 1,$$
$$\frac{d}{dy}\cosh y = \sinh y, \quad \frac{d}{dy}\sinh y = \cosh y.$$

One can show that the general solution to the ODE $Y'' - \mu^2 Y = 0$ can (also) be written as

$$Y = A\cosh(\mu y) + B\sinh(\mu y).$$

Using $\mu = \mu_n$ and $Y(0) = 0$, we find

$$\begin{aligned} Y(y) = Y_n(y) &= A_n \cosh(\mu_n y) + B_n \sinh(\mu_n y) \\ 0 = Y_n(0) &= A_n \cosh 0 + B_n \sinh 0 = A_n. \end{aligned}$$

This yields the separated solutions

$$u_n(x, y) = X_n(x) Y_n(y) = B_n \sin(\mu_n x) \sinh(\mu_n y),$$

and superposition gives the general solution

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x) \sinh(\mu_n y).$$

Finally, the top edge boundary condition requires that

$$f_2(x) = u(x, b) = \sum_{n=1}^{\infty} B_n \sinh(\mu_n b) \sin(\mu_n x).$$

Conclusion

Appealing to the formulae for sine series coefficients, we can now summarize our findings.

Theorem

If $f_2(x)$ is piecewise smooth, the solution to the Dirichlet problem

$$\begin{aligned} \Delta u &= 0, && 0 < x < a,\ 0 < y < b, \\ u(x,0) = 0,\ u(x,b) &= f_2(x), && 0 < x < a, \\ u(0,y) = u(a,y) &= 0, && 0 < y < b. \end{aligned}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x) \sinh(\mu_n y),$$

where $\mu_n = \dfrac{n\pi}{a}$ *and* $B_n = \dfrac{2}{a\sinh(\mu_n b)} \displaystyle\int_0^a f_2(x) \sin(\mu_n x)\, dx.$

Remark: If we know the sine series expansion for $f_2(x)$ on $[0, a]$, then we can use the relationship

$$B_n = \frac{1}{\sinh(\mu_n b)} \left(n\text{th sine coefficient of } f_2 \right).$$

Example

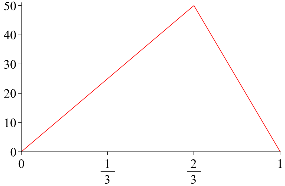
Solve the Dirichlet problem on the square $[0,1] \times [0,1]$, *subject to the boundary conditions*

$$\begin{aligned} &u(x,0) = 0,\ u(x,1) = f_2(x), && 0 < x < 1, \\ &u(0,y) = u(1,y) = 0, && 0 < y < 1. \end{aligned}$$

where

$$f_2(x) = \begin{cases} 75x & \textit{if } 0 \le x \le \frac{2}{3}, \\ 150(1-x) & \textit{if } \frac{2}{3} < x \le 1. \end{cases}$$

We have $a = b = 1$. The graph of $f_2(x)$ is:



According to exercise 2.4.17 (with $p = 1$, $a = 2/3$ and $h = 50$), the sine series for f_2 is:

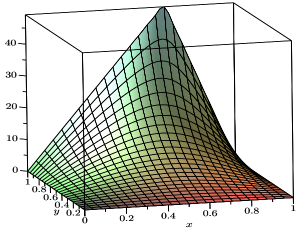
$$f_2(x) = \frac{450}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2} \sin(n\pi x).$$

Thus,

$$B_n = \frac{1}{\sinh(n\pi)}\left(\frac{450}{\pi^2}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2}\right) = \frac{450}{\pi^2}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\sinh(n\pi)},$$

and

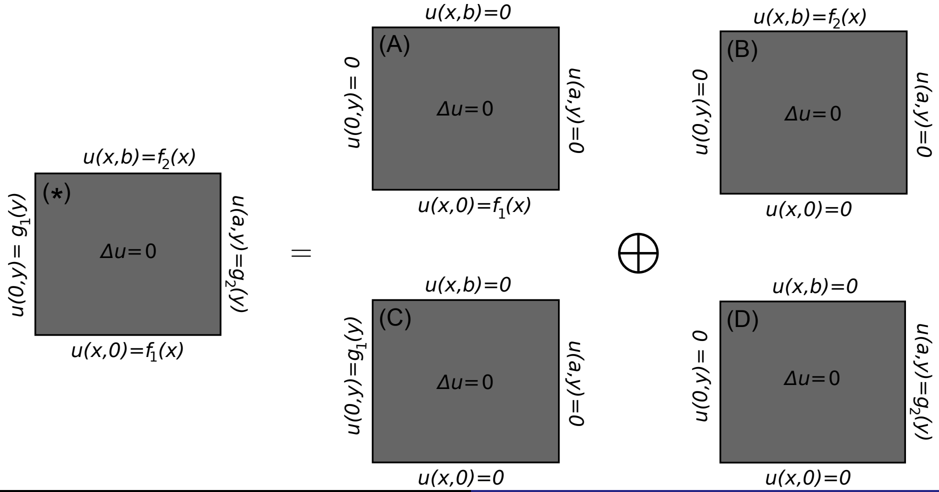
$$u(x,y) = \frac{450}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi y).$$



Solution of the Dirichlet problem on a rectangle

Complete solution

Recall:



Solution of the Dirichlet problem on a rectangle

Cases A and C

Separation of variables shows that the solution to (A) is

$$u_A(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right),$$

where

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_1(x)\sin\left(\frac{n\pi x}{a}\right)\,dx.$$

Likewise, the solution to (C) is

$$u_C(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(a-x)}{b}\right) \sin\left(\frac{n\pi y}{b}\right),$$

with

$$C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_1(y)\sin\left(\frac{n\pi y}{b}\right)\,dy.$$

Solution of the Dirichlet problem on a rectangle

Case D

And the solution to (D) is

$$u_D(x,y) = \sum_{n=1}^{\infty} D_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right),$$

where

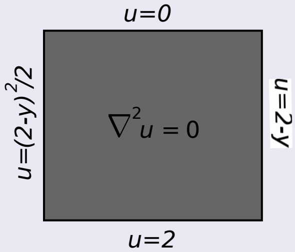
$$D_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_2(y) \sin\left(\frac{n\pi y}{b}\right)\, dy.$$

Remark: The coefficients in each case are just multiples of the Fourier sine coefficients of the nonzero boundary condition, e.g.

$$D_n = \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \left(n\text{th sine coefficient of } g_2 \text{ on } [0,b]\right).$$

Example

Solve the Dirichlet problem on $[0,1] \times [0,2]$ *with the following boundary conditions.*



We have $a = 1$, $b = 2$ and

$$f_1(x) = 2, \quad f_2(x) = 0, \quad g_1(y) = \frac{(2-y)^2}{2}, \quad g_2(y) = 2 - y.$$

It follows that $B_n = 0$ for all n, and the remaining coefficients we need are

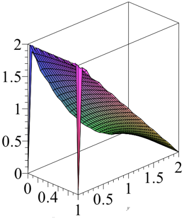
$$A_n = \frac{2}{1 \cdot \sinh\left(\frac{n\pi 2}{1}\right)} \int_0^1 2 \sin\left(\frac{n\pi x}{1}\right)\, dx = \frac{4(1 + (-1)^{n+1})}{n\pi \sinh(2n\pi)},$$

$$C_n = \frac{2}{2\sinh\left(\frac{n\pi 1}{2}\right)} \int_0^2 \frac{(2-y)^2}{2} \sin\left(\frac{n\pi y}{2}\right)\, dy = \frac{4(\pi^2 n^2 - 2 + 2(-1)^n)}{n^3\pi^3 \sinh\left(\frac{n\pi}{2}\right)},$$

$$D_n = \frac{2}{2\sinh\left(\frac{n\pi 1}{2}\right)} \int_0^2 (2-y) \sin\left(\frac{n\pi y}{2}\right)\, dy = \frac{4}{n\pi \sinh\left(\frac{n\pi}{2}\right)}.$$

The complete solution is thus

$$\begin{aligned} u(x,y) =& u_A(x,y) + u_C(x,y) + u_D(x,y) \\ =& \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{n\pi \sinh(2n\pi)} \sin(n\pi x) \sinh(n\pi(2-y)) \\ & + \sum_{n=1}^{\infty} \frac{4(n^2\pi^2 - 2 + 2(-1)^n)}{n^3\pi^3 \sinh\left(\frac{n\pi}{2}\right)} \sinh\left(\frac{n\pi(1-x)}{2}\right) \sin\left(\frac{n\pi y}{2}\right) \\ & + \sum_{n=1}^{\infty} \frac{4}{n\pi \sinh\left(\frac{n\pi}{2}\right)} \sinh\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right). \end{aligned}$$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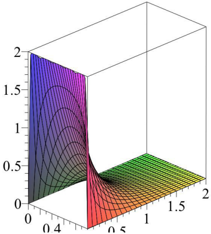
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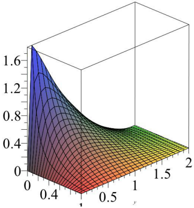
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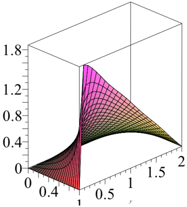
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