The two dimensional wave equation

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Partial Differential Equations
Lecture 14

Vibrating membranes

Goal: Model the motion of an ideal elastic membrane.

Set up: Assume the membrane at rest is a region of the *xy*-plane and let

$$u(x, y, t) = \begin{cases} \text{vertical deflection of membrane from equilib-} \\ \text{rium at position } (x, y) \text{ and time } t. \end{cases}$$

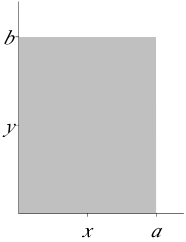
For a fixed t, the surface $z = u(x, y, t)$ gives the shape of the membrane at time t.

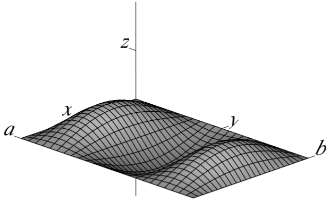
Under ideal assumptions (e.g. uniform membrane density, uniform tension, no resistance to motion, small deflection, etc.) one can show that u satisfies the *two dimensional wave equation*

$$u_{tt} = c^2 \Delta u = c^2(u_{xx} + u_{yy}).$$

Rectangular membranes

We assume the membrane lies over the rectangular region $R = [0, a] \times [0, b]$ and has fixed edges.





These facts are expressed by the *boundary conditions*

$$u(0, y, t) = u(a, y, t) = 0, \qquad 0 \le y \le b, \ t > 0,$$
$$u(x, 0, t) = u(x, b, t) = 0, \qquad 0 \le x \le a, \ t > 0.$$

We must also specify how the membrane is initially deformed and set into motion. This is done via the *initial conditions*

$$
\begin{aligned}
u(x,y,0) &= f(x,y), \qquad && (x,y) \in R, \\
u_t(x,y,0) &= g(x,y), \qquad && (x,y) \in R.
\end{aligned}
$$

New goal: solve the 2-D wave equation subject to the boundary and initial conditions just given.

As usual, one can:

- Use separation of variables to find separated solutions satisfying the homogeneous boundary conditions; and
- Use the principle of superposition to build up a series solution that satisfies the initial conditions as well.

Separation of variables

We seek nontrivial solutions of the form

$$u(x, y, t) = X(x)Y(y)T(t).$$

Plugging this into $u_{tt} = c^2(u_{xx} + u_{yy})$ and separating variables (as with the 2D heat equation) yields the separated system of ODEs and boundary conditions:

$$X'' - BX = 0, \ \ X(0) = X(a) = 0,$$

$$Y'' - CY = 0, \ \ Y(0) = Y(b) = 0,$$

$$T'' - c^2 AT = 0.$$

in which $A = B + C$. Notice that there are no boundary conditions on T.

We have already solved the two boundary value problems for X and Y. The nontrivial solutions are

$$X = X_m(x) = \sin(\mu_m x), \qquad \mu_m = \frac{m\pi}{a}, \qquad m \in \mathbb{N},$$
$$Y = Y_n(y) = \sin(\nu_n y), \qquad \nu_n = \frac{n\pi}{b}, \qquad n \in \mathbb{N},$$

with separation constants $B = -\mu_m^2$ and $C = -\nu_n^2$.

Since $T'' - c^2 AT = 0$, and $A = B + C = -\left(\mu_m^2 + \nu_n^2\right) < 0$,

$$T = T_{mn}(t) = B_{mn}\cos(\lambda_{mn}t) + B_{mn}^*\sin(\lambda_{mn}t),$$

where

$$\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

These are the **characteristic frequencies** of the membrane.

Normal modes

Assembling our results, we find that for any *pair* $m, n \in \mathbb{N}$ we have the *normal mode*

$$\begin{aligned} u_{mn}(x,y,t) &= X_m(x)Y_n(y)T_{mn}(t) \\ &= \sin(\mu_m x)\sin(\nu_n y)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right) \\ &= A_{mn}\sin(\mu_m x)\sin(\nu_n y)\cos(\lambda_{mn}t - \phi_{mn}) \end{aligned}$$

Remarks: Note that the normal modes:

- oscillate spatially with period $2\pi/\mu_m = 2a/m$ in the x-direction, and with period $2\pi/\nu_n = 2b/n$ in the y-direction;
- oscillate in time with frequency $\lambda_{mn}/2\pi$.

Notice that $\lambda_{mn}/2\pi$ *is not a multiple of any basic frequency.* So the general solution $u(x,y,t)$ will be oscillatory, but *not necessarily periodic (in time).*

Superposition and initial conditions

Superposition gives the general solution

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(\mu_m x)\sin(\nu_n y)\left(B_{mn}\cos(\lambda_{mn}t)+B_{mn}^{*}\sin(\lambda_{mn}t)\right).$$

The initial conditions will determine the coefficients B_{mn} and B_{mn}^{*}. Setting $t=0$ yields

$$f(x,y)=u(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

$$g(x,y)=u_t(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\lambda_{mn}B_{mn}^{*}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right).$$

These are again *double Fourier series* whose coefficients are given by double integrals.

Conclusion

Theorem

The solution to the vibrating membrane problem is given by
$u(x, y, t) =$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(\mu_m x)\sin(\nu_n y)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right)$$

where $\mu_m = \frac{m\pi}{a}$, $\nu_n = \frac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, *and*

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x, y)\sin(\mu_m x)\sin(\nu_n y)\,dy\,dx,$$

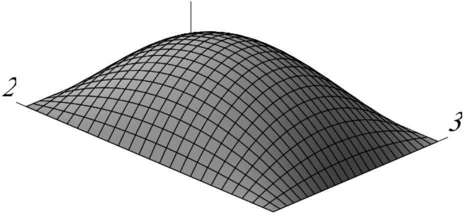
$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x, y)\sin(\mu_m x)\sin(\nu_n y)\,dy\,dx.$$

Example

A 2×3 *rectangular membrane has* $c = 6$*. If we deform it to have shape given by*

$$f(x,y) = xy(2-x)(3-y),$$

keep its edges fixed, and release it at $t = 0$*, find an expression that gives the shape of the membrane for* $t > 0$*.*



We must compute the coefficients B_{mn} and B_{mn}^*. Since $g(x,y)=0$ we immediately have

$$B_{mn}^* = 0.$$

We also have

$$\begin{aligned}
B_{mn} &= \frac{4}{2\cdot 3}\int_0^2\int_0^3 xy(2-x)(3-y)\sin\left(\frac{m\pi}{2}x\right)\,\sin\left(\frac{n\pi}{3}y\right)\,dy\,dx \\
&= \frac{2}{3}\int_0^2 x(2-x)\sin\left(\frac{m\pi}{2}x\right)\,dx\int_0^3 y(3-y)\sin\left(\frac{n\pi}{3}y\right)\,dy \\
&= \frac{2}{3}\left(\frac{16(1+(-1)^{m+1})}{\pi^3m^3}\right)\left(\frac{54(1+(-1)^{n+1})}{\pi^3n^3}\right) \\
&= \frac{576}{\pi^6}\frac{(1+(-1)^{m+1})(1+(-1)^{n+1})}{m^3n^3}.
\end{aligned}$$

The coefficients λ_{mn} are given by

$$\lambda_{mn} = c\sqrt{\mu_n^2 + \nu_n^2} = 6\pi\sqrt{\frac{m^2}{4} + \frac{n^2}{9}} = \pi\sqrt{9m^2 + 4n^2}.$$

Assembling all of these pieces yields

$$u(x, y, t) = \frac{576}{\pi^6} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{(1 + (-1)^{m+1})(1 + (-1)^{n+1})}{m^3 n^3} \sin\left(\frac{m\pi}{2} x\right) \right.$$
$$\left. \times \sin\left(\frac{n\pi}{3} y\right) \cos\left(\pi\sqrt{9m^2 + 4n^2}\, t\right)\right).$$

Example

Suppose in the previous example we also impose an initial velocity given by $g(x, y) = 8\sin 2\pi x$. Find an expression that gives the shape of the membrane for $t > 0$.

Since we have the same initial shape, B_{mn} don't change. We only need to find B^*_{mn} and add the appropriate terms to the previous solution.

Using λ_{mn} computed above, we have

$$\begin{aligned} B^*_{mn} &= \frac{4}{2 \cdot 3\pi\sqrt{9m^2 + 4n^2}} \int_0^2 \int_0^3 8\sin(2\pi x)\sin\left(\frac{m\pi}{2}x\right)\sin\left(\frac{n\pi}{3}y\right)\,dy\,dx \\ &= \frac{16}{3\pi\sqrt{9m^2 + 4n^2}} \int_0^2 \sin(2\pi x)\sin\left(\frac{m\pi}{2}x\right)\,dx \int_0^3 \sin\left(\frac{n\pi}{3}y\right)\,dy. \end{aligned}$$

The first integral is zero unless $m = 4$, i.e. $B^*_{mn} = 0$ for $m \neq 4$.

Evaluating the second integral, we have

$$B_{4n}^* = \frac{8}{3\pi\sqrt{36+n^2}} \frac{3(1+(-1)^{n+1})}{n\pi} = \frac{8(1+(-1)^{n+1})}{\pi^2 n\sqrt{36+n^2}}.$$

So the velocity dependent term of the solution is

$$\begin{aligned} u_2(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^* \sin(\mu_m x)\sin(\nu_n y)\sin(\lambda_{mn} t) \\ &= \frac{8\sin(2\pi x)}{\pi^2}\sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n\sqrt{36+n^2}} \sin\left(\frac{n\pi}{3}y\right) \sin\left(2\pi\sqrt{36+n^2}\,t\right). \end{aligned}$$

If we let $u_1(x,y,t)$ denote the solution to the first example, the complete solution here is

$$u(x,y,t) = u_1(x,y,t) + u_2(x,y,t).$$

The vibrating circular membrane

Goal: Model the motion of an elastic membrane stretched over a circular frame of radius a.

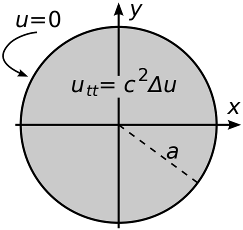
Set-up: Center the membrane at the origin in the xy-plane and let

$$u(r,\theta,t) = \begin{cases} \text{deflection of membrane from equilibrium at} \\ \text{polar position } (r,\theta) \text{ and time } t. \end{cases}$$

Under ideal assumptions:

$$u_{tt} = c^2 \Delta u = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right), \quad 0 < r < a, \ 0 < \theta < 2\pi, \ t > 0,$$

$$u(a,\theta,t) = 0, \quad 0 \le \theta \le 2\pi, \quad t > 0.$$



Separation of variables

Setting $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ leads to the separated boundary value problems

$$r^2R'' + rR' + \left(\lambda^2 r^2 - \mu^2\right)R = 0, \quad R(0+) \text{ finite}, \quad R(a) = 0,$$
$$\Theta'' + \mu^2\Theta = 0, \quad \Theta \ 2\pi\text{-periodic},$$
$$T'' + c^2\lambda^2 T = 0.$$

We have already seen that the solutions to the Θ problem are

$$\Theta(\theta) = \Theta_m(\theta) = A\cos(m\theta) + B\sin(m\theta), \quad \mu = m \in \mathbb{N}_0.$$

So, for each $m \in \mathbb{N}_0$ it remains to solve the ODE boundary value problem

$$r^2R'' + rR' + \left(\lambda^2 r^2 - m^2\right)R = 0, \quad R(0+) \text{ finite}, \quad R(a) = 0.$$

Solving for R

Case 1: $\lambda = 0$. This is an Euler equation, and the only solution to the BVP is $R \equiv 0$ (HW).

Case 2: $\lambda > 0$. The changes of variables $R(r) = y(x)$, $x = \lambda r$ lead to

$$\underbrace{x^2y'' + xy' + (x^2 - m^2)y = 0}_{\text{Bessel's equation of order } m}, \quad y(0+) \text{ finite}, \quad y(\lambda a) = 0.$$

Remarks.

- The solutions to Bessel's equation have been well-studied.
- The standard normalized independent solutions are known as the *Bessel functions of the first and second kind.*

Bessel's equation

Given $p \geq 0$, the ordinary differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0, \qquad x > 0$$

is known as *Bessel's equation of order* p. Using the *Method of Frobenius* one arrives at the series solution

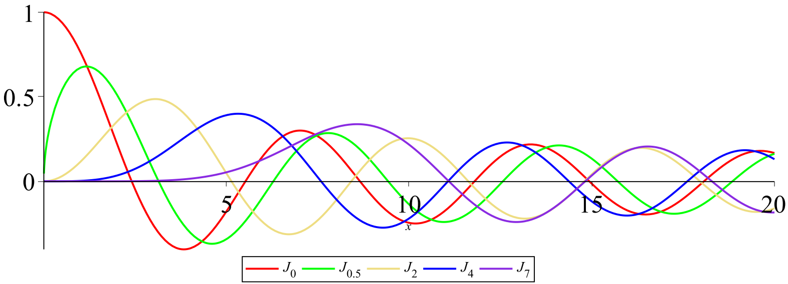
$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p},$$

which is known as the *Bessel function of the first kind of order* p.

Here Γ denotes the *Gamma function*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt \qquad (s > 0).$$

Graphs of Bessel functions of the first kind



In Maple, the functions $J_p(x)$ can be invoked by the command

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BesselJ(p,x)
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Properties of Bessel functions of the first kind

- $J_0(0) = 1$ and $J_p(0) = 0$ for $p > 0$.
- The values of J_p always lie between 1 and -1.
- J_p has infinitely many positive zeros, which we denote by

$$0 < \alpha_{p1} < \alpha_{p2} < \alpha_{p3} < \cdots$$

- J_p is oscillatory and tends to zero as $x \to \infty$. More precisely,

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right).$$

- $\lim_{n\to\infty} |\alpha_{pn} - \alpha_{p,n+1}| = \pi$.

- For $0 < p < 1$, the graph of J_p has a vertical tangent line at $x = 0$.

- For $1 < p$, the graph of J_p has a horizontal tangent line at $x = 0$, and the graph is initially "flat."

- For some values of p, the Bessel functions of the first kind can be expressed in terms of familiar functions, e.g.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$
$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right).$$

Remarks

- Frobenius' method yields a second linearly independent solution y_2 of Bessel's equation.
- Although the exact form of y_2 depends on the value of p, it is not hard to argue that in any case $\lim_{x \to 0^+} |y_2| = \infty$.
- Since $\lim_{x \to 0^+} J_p(x)$ is finite, it follows that *any* linearly independent solution $Y_p(x)$ must also satisfy

$$\lim_{x \to 0^+} |Y_p(x)| = \infty.$$

.

- The standard normalization of Y_p is called the *Bessel function of the second kind*. We won't explicitly need it.

Differentiation identities

Using the series definition of $J_p(x)$, one can show that:

$$\begin{aligned} \frac{d}{dx}\left(x^p J_p(x)\right) &= x^p J_{p-1}(x), \\ \frac{d}{dx}\left(x^{-p} J_p(x)\right) &= -x^{-p} J_{p+1}(x). \end{aligned} \tag{1}$$

The product rule and cancellation lead to

$$\begin{aligned} xJ_p'(x) + pJ_p(x) &= xJ_{p-1}(x), \\ xJ_p'(x) - pJ_p(x) &= -xJ_{p+1}(x). \end{aligned}$$

Addition and subtraction of these identities then yield

$$\begin{aligned} J_{p-1}(x) - J_{p+1}(x) &= 2J_p'(x), \\ J_{p-1}(x) + J_{p+1}(x) &= \frac{2p}{x} J_p(x). \end{aligned}$$

Integration identities

Integration of the differentiation identities (1) gives

$$\int x^{p+1} J_p(x)\,dx = x^{p+1} J_{p+1}(x) + C$$
$$\int x^{-p+1} J_p(x)\,dx = -x^{-p+1} J_{p-1}(x) + C.$$

- Exercises 4.2.12 and 4.3.9 give similar identities.
- Identities such as these can be used to evaluate certain integrals of the form

$$\int_0^a f(r) J_m(\lambda_{mn} r) r\,dr,$$

which will occur frequently in later work.

Example

Evaluate

$$\int x^{p+5} J_p(x)\, dx.$$

We integrate by parts, first taking

$$u = x^4 \qquad dv = x^{p+1} J_p(x)\, dx$$
$$du = 4x^3\, dx \qquad v = x^{p+1} J_{p+1}(x),$$

which gives

$$\int x^{p+5} J_p(x)\, dx = x^{p+5} J_{p+1}(x) - 4 \int x^{p+4} J_{p+1}(x)\, dx.$$

Now integrate by parts again with

$$u = x^2 \qquad dv = x^{p+2} J_{p+1}(x)\,dx$$
$$du = 2x\,dx \qquad v = x^{p+2} J_{p+2}(x),$$

to get

$$\begin{aligned}\int x^{p+5} J_p(x)\,dx &= x^{p+5} J_{p+1}(x) - 4\int x^{p+4} J_{p+1}(x)\,dx \\ &= x^{p+5} J_{p+1}(x) - 4\left(x^{p+4} J_{p+2}(x) - 2\int x^{p+3} J_{p+2}(x)\,dx\right) \\ &= x^{p+5} J_{p+1}(x) - 4x^{p+4} J_{p+2}(x) + 8x^{p+3} J_{p+3}(x) + C.\end{aligned}$$