Complete Solution of the Wave Equation on a Disk

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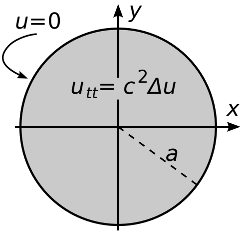
Partial Differential Equations
Lecture 15

Return of the vibrating circular membrane

Recall the vibrating circular membrane problem:



Separation of variables in polar coordinates led to the ODE boundary value problem

$$r^2 R'' + rR' + \left(\lambda^2 r^2 - \mu^2\right) R = 0, \quad R(0+) \text{ finite}, \quad R(a) = 0,$$

$$\Theta'' + \mu^2 \Theta = 0, \quad \Theta \;\; 2\pi\text{-periodic},$$

$$T'' + c^2 \lambda^2 T = 0.$$

We have seen that the solutions to the Θ problem are

$$\Theta(\theta) = \Theta_m(\theta) = A\cos(m\theta) + B\sin(m\theta), \quad \mu = m \in \mathbb{N}_0.$$

In the HW you were asked to show that $\lambda = 0$ implies $R \equiv 0$, so we are faced with solving the *parametric Bessel equation*

$$r^2R'' + rR' + (\lambda^2 r^2 - m^2)R = 0 \quad (\lambda > 0) \tag{1}$$

subject to the boundary conditions

$$R(0+) \text{ finite}, \quad R(a) = 0.$$

If we let $x = \lambda r$, then the chain rule implies

$$R' = \frac{dR}{dr} = \frac{dR}{dx}\frac{dx}{dr} = \lambda\dot{R},$$
$$R'' = \frac{dR'}{dr} = \lambda\frac{d\dot{R}}{dr} = \lambda\frac{d\dot{R}}{dx}\frac{dx}{dr} = \lambda^2\ddot{R}.$$

Hence (1) becomes

$$x^2\ddot{R} + x\dot{R} + (x^2 - m^2)R = 0,$$

which is Bessel's equation of order m.

It follows that

$$R = c_1 J_m(x) + c_2 Y_m(x) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r),$$

where $J_m(x)$ and $Y_m(x)$ are the Bessel functions (of order m) of the first and second kind (respectively).

Because $J_m(0)$ is finite while $\lim_{x \to 0^+} |Y_m(x)| = \infty$, we find that

$$R(0+) \text{ finite} \quad \Rightarrow \quad c_2 = 0 \quad \Rightarrow \quad R = c_1 J_m.$$

$$R(a) = 0 \;\Rightarrow\; R(a) = c_1 J_m(\lambda a) = 0 \;\underset{c_1 \neq 0}{\Rightarrow}\; J_m(\lambda a) = 0$$

$$\Rightarrow\; \lambda a = \alpha_{mn}, \quad n \in \mathbb{N}$$

$$\Rightarrow\; \lambda = \lambda_{mn} = \frac{\alpha_{mn}}{a}, \quad n \in \mathbb{N}.$$

Choosing $c_1 = 1$, we find that

$$R(r) = R_{mn}(r) = J_m(\lambda_{mn} r) = J_m\left(\frac{\alpha_{mn} r}{a}\right) \quad m \in \mathbb{N}_0, \;\; n \in \mathbb{N}.$$

For each fixed $m \in \mathbb{N}_0$ these are analogous to the spatial factors

$$X(x) = X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \;\; n \in \mathbb{N},$$

of the normal modes in the 1D vibrating string problem.

Normal modes of the vibrating circular membrane

Returning to T (which solves $T'' + c^2\lambda^2 T = 0$), we finally find

$$T(t) = T_{mn}(t) = C\cos(c\lambda_{mn}t) + D\sin(c\lambda_{mn}t).$$

So we arrive at the normal modes for the vibrating circular membrane:

$$u_{mn}(r,\theta,t) = R_{mn}(r)\Theta_m(\theta)T_{mn}(t) =$$

$$J_m(\lambda_{mn}r)\,(A\cos(m\theta) + B\sin(m\theta))\,(C\cos(c\lambda_{mn}t) + D\sin(c\lambda_{mn}t)),$$

for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, where $\lambda_{mn} = \alpha_{mn}/a$, $a > 0$ is the radius of the membrane, and

$$\alpha_{m1} < \alpha_{m2} < \alpha_{m3} < \cdots$$

are the positive zeros of $J_m(x)$.

Motion of one circular normal mode

The coefficients A, B, C, D depend on m, n and will be determined by the initial conditions imposed on the membrane.

Note that, up to scaling, rotation and a phase shift in time, every mode has the form

$$u(r, \theta, t) = J_m(\lambda_{mn} r) \cos(m\theta) \cos(c\lambda_{mn} t).$$

These oscillate:

- spatially in the radial (r) and angular (θ) directions;
- in time with frequency $\dfrac{c\lambda_{mn}}{2\pi}$.

Since $\lambda_{mn} = \alpha_{mn}/a$, they become more oscillatory as m and n increase.

The general (series) solution

First we we split the general normal mode

$$J_m(\lambda_{mn} r)\,(A\cos(m\theta) + B\sin(m\theta))\,(C\cos(c\lambda_{mn} t) + D\sin(c\lambda_{mn} t)),$$

in two. For convenience we set

$$u_{mn}(r,\theta,t) = J_m(\lambda_{mn} r)\,(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta))\cos(c\lambda_{mn} t),$$

$$u^*_{mn}(r,\theta,t) = J_m(\lambda_{mn} r)\,(a^*_{mn}\cos(m\theta) + b^*_{mn}\sin(m\theta))\sin(c\lambda_{mn} t),$$

and use superposition to construct the general solution

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} u_{mn}(r,\theta,t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} u^*_{mn}(r,\theta,t).$$

Imposing the initial conditions

In order to completely determine the shape of the membrane at any time we must specify the *initial conditions*

$$u(r,\theta,0) = f(r,\theta),\ \ 0 \le r \le a,\ \ 0 \le \theta \le 2\pi\ \ \text{(shape)},$$
$$u_t(r,\theta,0) = g(r,\theta),\ \ 0 \le r \le a,\ \ 0 \le \theta \le 2\pi\ \ \text{(velocity)}.$$

Setting $t = 0$ in the general solution, we find that this requires

$$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\right)$$
$$g(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\lambda_{mn} J_m(\lambda_{mn} r)\left(a^*_{mn}\cos(m\theta) + b^*_{mn}\sin(m\theta)\right)$$

which are called *Fourier-Bessel expansions*.

Othogonality of Bessel functions

We will see later that the functions $R_{mn}(r) = J_m(\lambda_{mn} r)$ are orthogonal relative to the *weighted inner product*

$$\langle f, g \rangle = \int_0^a f(r) g(r) r \, dr.$$

That is,

$$\langle R_{mn}, R_{mk} \rangle = \int_0^a J_m(\lambda_{mn} r) \, J_m(\lambda_{mk} r) \, r \, dr = 0 \quad \text{if } n \neq k.$$

In addition, it can also be shown that

$$\langle R_{mn}, R_{mn} \rangle = \int_0^a J_m^2(\lambda_{mn} r) r \, dr = \frac{a^2}{2} J_{m+1}^2(\alpha_{mn}).$$

Using the orthogonality relations for Bessel and trigonometric functions, one obtains:

Theorem

The functions

$$\phi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\cos(m\theta),$$
$$\psi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\sin(m\theta),$$

($m \in \mathbb{N}_0$, $n \in \mathbb{N}$) form a (complete) orthogonal set of functions relative to the (polar coordinate) inner product

$$\langle f, g\rangle = \int_0^{2\pi}\int_0^a f(r,\theta)g(r,\theta)r\,dr\,d\theta.$$

That is, $\langle \phi_{mn}, \phi_{jk}\rangle = \langle \psi_{mn}, \psi_{jk}\rangle = 0$ *for* $(m,n) \neq (j,k)$ *and* $\langle \phi_{mn}, \psi_{jk}\rangle = 0$ *for all* (m,n) *and* (j,k).

Since our initial membrane shape condition is

$$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(a_{mn}\phi_{mn}(r,\theta) + b_{mn}\psi_{mn}(r,\theta)\right),$$

the usual orthogonality argument gives

$$a_{mn} = \frac{\langle f, \phi_{mn}\rangle}{\langle \phi_{mn}, \phi_{mn}\rangle} = \frac{\displaystyle\int_0^{2\pi}\int_0^a f(r,\theta)J_m(\lambda_{mn}r)\cos(m\theta)\, r\, dr\, d\theta}{\displaystyle\int_0^{2\pi}\int_0^a J_m^2(\lambda_{mn}r)\, \cos^2(m\theta)\, r\, dr\, d\theta},$$

$$b_{mn} = \frac{\langle f, \psi_{mn}\rangle}{\langle \psi_{mn}, \psi_{mn}\rangle} = \frac{\displaystyle\int_0^{2\pi}\int_0^a f(r,\theta)J_m(\lambda_{mn}r)\sin(m\theta)\, r\, dr\, d\theta}{\displaystyle\int_0^{2\pi}\int_0^a J_m^2(\lambda_{mn}r)\, \sin^2(m\theta)\, r\, dr\, d\theta},$$

for $m \geq 0$, $n \geq 1$.

The integrals in the denominators can be evaluated explicitly:

$$\int_0^{2\pi}\int_0^a J_m^2(\lambda_{mn} r)\,\cos^2(m\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\cos^2(m\theta)\,d\theta\,\int_0^a J_m^2(\lambda_{mn} r)\,r\,dr = \begin{cases} \pi a^2 J_1^2(\alpha_{0n}) & \text{if } m=0, \\ \dfrac{\pi a^2}{2} J_{m+1}^2(\alpha_{mn}) & \text{if } m\geq 1; \end{cases}$$

and likewise

$$\int_0^{2\pi}\int_0^a J_m^2(\lambda_{mn} r)\,\sin^2(m\theta)\,r\,dr\,d\theta = \frac{\pi a^2}{2} J_{m+1}^2(\alpha_{mn}),$$

for $m \geq 1$.

Integral formulae for a_{mn} and b_{mn}

We conclude that

$$a_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^a f(r,\theta)\, J_0(\lambda_{0n} r)\, r\, dr\, d\theta,$$

$$a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r,\theta)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta,$$

$$b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r,\theta)\, J_m(\lambda_{mn} r)\, \sin(m\theta)\, r\, dr\, d\theta,$$

for $m, n \in \mathbb{N}$. Finally, recall the initial velocity condition

$$g(r,\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(c\lambda_{mn} a_{mn}^* \phi_{mn}(r,\theta) + c\lambda_{mn} b_{mn}^* \psi_{mn}(r,\theta) \right).$$

Integral formulae for a^*_{mn} and b^*_{mn}

The same line of reasoning as above yields

$$a^*_{0n} = \frac{1}{\pi c \alpha_{0n} a J_1^2(\alpha_{0n})} \int_0^{2\pi}\int_0^a g(r,\theta)\, J_0(\lambda_{0n} r)\, r\, dr\, d\theta,$$

$$a^*_{mn} = \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^a g(r,\theta)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta,$$

$$b^*_{mn} = \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^a g(r,\theta)\, J_m(\lambda_{mn} r)\, \sin(m\theta)\, r\, dr\, d\theta,$$

for $m, n \in \mathbb{N}$.

This (essentially) completes the statement of the general solution to the vibrating circular membrane problem

Remark

Since $\cos 0 = 1$ and $\sin 0 = 0$ we have

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\right)\cos(c\lambda_{mn} t)$$

$$= \underbrace{\sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r)\cos(c\lambda_{0n} t)}_{m=0} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\text{as above})$$

- Note that there are really *no* b_{0n} coefficients.
- This is the “true form” of the first series in the solution.

Analogous comments hold for the second series.

Remark

If $f(r,\theta) = f(r)$ (i.e. f is *radially symmetric*), then for $m \neq 0$

$$a_{mn} = (\cdots) \int_0^{2\pi} \int_0^a f(r)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta$$
$$= (\cdots) \int_0^a \cdots\, dr \underbrace{\int_0^{2\pi} \cos(m\theta)\, d\theta}_{0} = 0,$$

and $b_{mn} = 0$, too. That is, there are *only* a_{0n} terms.

Likewise, if g is radially symmetric, then for $m \neq 0$

$$a^*_{mn} = b^*_{mn} = 0,$$

and there are *only* a^*_{0n} terms.

Example

Solve the vibrating membrane problem with $a = c = 1$ and initial conditions

$$f(r,\theta) = 1 - r^4, \qquad g(r,\theta) = 0.$$

Because $g(r,\theta) = 0$, we immediately find that $a^*_{mn} = b^*_{mn} = 0$ for all m and n.

Because f is radially symmetric, we only need to compute a_{0n}. Since $a = 1$, $\lambda_{mn} = \alpha_{mn}$, so

$$\begin{aligned} a_{0n} &= \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^1 f(r) J_0(\alpha_{0n} r) r \, dr d\theta \\ &= \frac{2}{J_1^2(\alpha_{0n})} \underbrace{\int_0^1 (1 - r^4) J_0(\alpha_{0n} r) r \, dr}_{\text{substitute } x = \alpha_{0n} r} \end{aligned}$$

$$= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \int_0^{\alpha_{0n}} \left(1 - \frac{x^4}{\alpha_{0n}^4}\right) J_0(x)x\,dx$$

$$= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \left(\underbrace{\int_0^{\alpha_{0n}} xJ_0(x)\,dx}_{A} - \frac{1}{\alpha_{0n}^4} \underbrace{\int_0^{\alpha_{0n}} x^5 J_0(x)\,dx}_{B} \right).$$

According to earlier results

$$A = \int_0^{\alpha_{0n}} xJ_0(x)\,dx = xJ_1(x)\Big|_0^{\alpha_{0n}} = \alpha_{0n}J_1(\alpha_{0n}),$$

$$B = \int_0^{\alpha_{0n}} x^5 J_0(x)\,dx = x^5J_1(x) - 4x^4J_2(x) + 8x^3J_3(x)\Big|_0^{\alpha_{0n}}$$

$$= \alpha_{0n}^5 J_1(\alpha_{0n}) - 4\alpha_{0n}^4 J_2(\alpha_{0n}) + 8\alpha_{0n}^3 J_3(\alpha_{0n}).$$

It follows that

$$a_{0n} = \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})}\left(A - \frac{1}{\alpha_{0n}^4}B\right) = \frac{8\left(\alpha_{0n}J_2(\alpha_{0n}) - 2J_3(\alpha_{0n})\right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})},$$

so that finally

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \frac{8\left(\alpha_{0n}J_2(\alpha_{0n}) - 2J_3(\alpha_{0n})\right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})} J_0(\alpha_{0n}r)\cos(\alpha_{0n}t).$$

Remark: This solution can easily be implemented in Maple, since the command

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BesselJZeros(m,n)
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will compute α_{mn} numerically.

A non-symmetric example

Example

Solve the vibrating membrane problem with $a = c = 1$ and initial conditions

$$f(r,\theta) = r(1 - r^4)\cos\theta, \qquad g(r,\theta) = 0.$$

Since $g \equiv 0$, $a^*_{mn} = b^*_{mn} = 0$ for all m, n. We also have

$$\begin{aligned} b_{mn} &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \int_0^{2\pi}\int_0^1 r(1-r^4)\cos\theta J_m(\alpha_{mn} r)\sin(m\theta) r\,dr d\theta \\ &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos\theta \sin(m\theta)\,d\theta}_{0} \int_0^1 r(1-r^4) J_m(\alpha_{mn} r) r\,dr \\ &= 0 \qquad \text{for all } m, n. \end{aligned}$$

Additionally,

$$\begin{aligned} a_{0n} &= \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi}\int_0^1 r(1-r^4)\ \cos\theta J_0(\alpha_{0n} r) r\, dr d\theta \\ &= \frac{1}{\pi J_1^2(\alpha_{0n})} \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} \int_0^1 r(1-r^4) J_0(\alpha_{0n} r) r\, dr \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} a_{mn} &= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^1 r(1-r^4)\ \cos\theta J_m(\alpha_{mn} r) \cos(m\theta) r\, dr d\theta \\ &= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos\theta\ \cos(m\theta)\, d\theta}_{A} \int_0^1 r(1-r^4) J_m(\alpha_{mn} r) r\, dr. \end{aligned}$$

The integral A is zero unless $m = 1$, in which case it's equal to π. In this case

$$a_{1n} = \frac{2}{J_2^2(\alpha_{1n})} \int_0^1 r(1 - r^4) J_1(\alpha_{1n} r) r \, dr$$
$$= \frac{2}{J_2^2(\alpha_{1n})} \left(\int_0^1 r^2 J_1(\alpha_{1n} r) \, dr \; - \int_0^1 r^6 J_1(\alpha_{1n} r) \, dr \right).$$

Substituting $x = \alpha_{1n} r$ and proceeding as before one can show

$$\int_0^1 r^2 J_1(\alpha_{1n} r) \, dr = \frac{J_2(\alpha_{1n})}{\alpha_{1n}},$$

$$\int_0^1 r^6 J_1(\alpha_{1n} r) \, dr = \frac{J_2(\alpha_{1n})}{\alpha_{1n}} - \frac{4 J_3(\alpha_{1n})}{\alpha_{1n}^2} + \frac{8 J_4(\alpha_{1n})}{\alpha_{1n}^3}.$$

Assembling these formulae gives

$$a_{1n} = \frac{2}{J_2^2(\alpha_{1n})}\left(\frac{4J_3(\alpha_{1n})}{\alpha_{1n}^2} - \frac{8J_4(\alpha_{1n})}{\alpha_{1n}^3}\right) = \frac{8\left(\alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n})\right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})}.$$

Since all the other coefficients are zero,

$$u(r,\theta,t) = \cos\theta \sum_{n=1}^{\infty} \frac{8\left(\alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n})\right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})} J_1(\alpha_{1n} r)\cos(\alpha_{1n} t).$$

Remark: In general, one should **not** expect the solution to reduce to a single series.

A "complicated" example

Example

Solve the vibrating membrane problem with $a = 2$, $c = 1$ and initial conditions

$$f(r,\theta) = 0, \qquad g(r,\theta) = r^2(2-r)\,\sin^8\left(\frac{\theta}{2}\right).$$

Since $f \equiv 0$, $a_{mn} = 0$, $b_{mn} = 0$. We also have

$$b^*_{mn} = (\cdots)\int_0^2 (\cdots)\,dr \int_0^{2\pi} \underbrace{\sin^8\left(\frac{\theta}{2}\right)\sin(m\theta)}_{\text{odd, } 2\pi\text{-periodic}}\,d\theta = 0,$$

$$a^*_{0n} = \frac{1}{\pi\alpha_{0n}2J_1^2(\alpha_{0n})}\underbrace{\int_0^{2\pi}\sin^8\left(\frac{\theta}{2}\right)\,d\theta}_{35\pi/64 \text{ (Maple)}}\ \underbrace{\int_0^2 r^2(2-r)J_0(\lambda_{0n}r)r\,dr}_{?},$$

and

$$a^*_{mn} = \frac{2}{\pi\alpha_{mn} 2J^2_{m+1}(\alpha_{mn})} \underbrace{\int_0^{2\pi} \sin^8\left(\frac{\theta}{2}\right)\cos(m\theta)\,d\theta}_{0 \text{ if } m\geq 5 \text{ (Maple)}} \cdot \underbrace{\int_0^2 r^2(2-r)J_m(\lambda_{mn}r)r\,dr}_{?}.$$

The solution therefore can be written

$$u(r,\theta,t) = \sum_{m=0}^{4}\sum_{n=1}^{\infty} a^*_{mn} J_m(\lambda_{mn}r)\cos(m\theta)\sin(\lambda_{mn}t),$$

although the (?) integrals are not amenable to evaluation by hand.