The Fourier Transform Method

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Partial Differential Equations
Lecture 17

Recall

The Fourier transform

The *Fourier transform* of a piecewise smooth $f \in L^1(\mathbb{R})$ is

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx,$$

and f can be recovered from $\hat{f}$ via the *inverse Fourier transform*

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega.$$

Remarks:

- See Appendix B1 for a table of Fourier transform pairs.
- The Fourier transform can help solve boundary value problems with *unbounded* domains.

Fourier transforms of two-variable functions

If $u(x,t)$ is defined for $-\infty < x < \infty$, we define its *Fourier transform in* x to be

$$\hat{u}(\omega,t) = \mathcal{F}(u(x,t))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx.$$

Because the Fourier transform treats t as a constant, we have

$$\mathcal{F}\left(\frac{\partial^n u}{\partial x^n}\right) = (i\omega)^n \mathcal{F}(u) = (i\omega)^n \hat{u}$$

and

$$\begin{aligned}\mathcal{F}\left(\frac{\partial^n u}{\partial t^n}\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^n u}{\partial t^n}(x,t)e^{-i\omega x}\,dx \\ &= \frac{\partial^n}{\partial t^n}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx\right) = \frac{\partial^n}{\partial t^n}\mathcal{F}(u) = \frac{\partial^n \hat{u}}{\partial t^n}.\end{aligned}$$

Example

Solve the 1-D heat equation on an infinite rod,

$$u_t = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x).$$

We take the Fourier transform (in x) on both sides to get

$$\hat{u}_t = c^2 (i\omega)^2 \hat{u} = -c^2 \omega^2 \hat{u}$$
$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

Since there is only a t derivative, we solve as though ω were a constant:

$$\widehat{u}(\omega, t) = A(\omega) e^{-c^2 \omega^2 t} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega).$$

To solve for u, we invert the Fourier transform, obtaining

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t) e^{i\omega x}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x}\, d\omega. \end{aligned}$$

Remarks.

- This expresses the solution in terms of the *Fourier transform* of the initial temperature distribution $f(x)$.
- We can obtain an (integral) expression for the solution directly in terms of f by instead recognizing the presence of a convolution, prior to Fourier inversion.

The heat kernel

The function

$$g_t(x) = \frac{1}{c\sqrt{2t}} e^{-x^2/(4c^2t)}$$

is called the *heat kernel*. We can use earlier results to deduce that

$$\widehat{g_t}(\omega) = e^{-c^2\omega^2 t},$$

and hence the solution above can also be written

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{-c^2\omega^2 t} = \widehat{f}(\omega)\widehat{g_t}(\omega) = \widehat{f * g_t}(\omega).$$

Applying $\mathcal{F}^{-1}$ to both sides this means that

$$\begin{aligned} u(x,t) = (f * g_t)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) g_t(x-s)\, ds \\ &= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/4c^2 t}\, ds. \end{aligned}$$

Example

Solve the boundary value problem

$$u_t = t u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x),$$

which models the temperature in an infinitely long rod with variable thermal diffusivity.

Taking the Fourier transform (in x) on both sides yields

$$\hat{u}_t = t(i\omega)^2 \hat{u} = -t\omega^2 \hat{u},$$
$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

The ODE in t is separable, with solution

$$\hat{u}(\omega, t) = A(\omega) e^{-t^2\omega^2/2} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega).$$

As before, Fourier inversion gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-t^2\omega^2/2} e^{i\omega x}\, d\omega.$$

In comparison with the preceding example, this decays more rapidly as t increases. This is is physically reasonable, since the thermal diffusivity is increasing with t.

Remark: Notice that this is the solution of the previous example, with $t^2/2$ replacing $c^2 t$. Using the earlier remark, this means

$$u(x,t) = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/2t^2}\, ds.$$

Example

Solve the third order mixed derivative boundary value problem

$$u_{tt} = u_{xxt}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Taking the Fourier transform (in x) on both sides yields

$$\hat{u}_{tt} = (i\omega)^2 \hat{u}_t = -\omega^2 \hat{u}_t,$$
$$\hat{u}(\omega,0) = \hat{f}(\omega), \quad \hat{u}_t(\omega,0) = \hat{g}(\omega)$$

Solving the ODE it t for $\hat{u}_t$ gives

$$\hat{u}_t(\omega,t) = A(\omega)e^{-\omega^2 t} \quad \Rightarrow \quad \hat{u}(\omega,t) = -\frac{A(\omega)}{\omega^2}e^{-\omega^2 t} + B(\omega)$$
$$= A(\omega)e^{-\omega^2 t} + B(\omega).$$

Imposing the initial conditions we find that

$$\begin{aligned} \hat{f}(\omega) &= \hat{u}(\omega,0) = A(\omega) + B(\omega) \\ \hat{g}(\omega) &= \hat{u}_t(\omega,0) = -\omega^2 A(\omega) \end{aligned} \quad \Rightarrow \quad \begin{aligned} A(\omega) &= \frac{-\hat{g}(\omega)}{\omega^2} \\ B(\omega) &= \hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2}. \end{aligned}$$

Plugging these into $\hat{u}$ and applying Fourier inversion yields

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{-\hat{g}(\omega)}{\omega^2} e^{-\omega^2 t} + \hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2} \right) e^{i\omega x}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2}(1 - e^{-\omega^2 t}) \right) e^{i\omega x}\, d\omega \\ &= f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(\omega)}{\omega^2}(1 - e^{-\omega^2 t}) e^{i\omega x}\, d\omega. \end{aligned}$$

Example

Solve the boundary value problem

$$t^2 u_x - u_t = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x),$$

and express the solution explicitly in terms of f.

Taking the Fourier transform (in x) on both sides yields

$$t^2(i\omega)\hat{u} - \hat{u}_t = 0,$$
$$\hat{u}(\omega,0) = \hat{f}(\omega).$$

The ODE in t is separable, with solution

$$\hat{u}(\omega,t) = A(\omega)e^{it^3\omega/3} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega,0) = A(\omega).$$

Using Fourier inversion leads to

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{it^3\omega/3} e^{i\omega x}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega(x+t^3/3)}\, d\omega \\ &= f\left(x + \frac{t^3}{3}\right). \end{aligned}$$

Remark: This particular problem is amenable to the *method of characteristics*, although the Fourier transform method may seem somewhat more straightforward.

Example

Solve the Dirichlet problem in the upper half-plane

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0,$$
$$u(x,0) = f(x),$$

which models the steady state temperature in a semi-infinite plate.

Taking the Fourier transform (in x) on both sides yields

$$(i\omega)^2 \hat{u} + \hat{u}_{yy} = \hat{u}_{yy} - \omega^2 \hat{u} = 0,$$
$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

The ODE in y has characteristic equation

$$r^2 - \omega^2 = 0 \quad \Rightarrow r = \pm\omega \quad \Rightarrow \quad \hat{u}(\omega, y) = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}.$$

We now require that $\hat{u}(\omega, y)$ remain bounded as $y \to \infty$. Consequently,

$$\left.\begin{array}{lcl} \omega > 0 & \Rightarrow & A(\omega) = 0 \\ \omega < 0 & \Rightarrow & B(\omega) = 0 \end{array}\right\} \Rightarrow \quad \hat{u}(\omega, y) = C(\omega)e^{-y|\omega|}$$

$$\Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = C(\omega)$$

Now recall that (for $a > 0$)

$$\left.\begin{array}{l} \mathcal{F}(e^{-|x|}) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{1+\omega^2} \\[2ex] \mathcal{F}(g(ax)) = \dfrac{1}{a}\hat{g}\left(\dfrac{\omega}{a}\right) \end{array}\right\} \Rightarrow \quad \mathcal{F}(e^{-a|x|}) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}.$$

Since

$$\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x),$$

applying $\mathcal{F}^{-1}$ to both sides, we have

$$e^{-a|x|} = \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}\right)(-x)$$

$$\Rightarrow \quad \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}\right)(x) = e^{-a|-x|} = e^{-a|x|}$$

$$\Rightarrow \quad e^{-y|\omega|} = \mathcal{F}\underbrace{\left(\sqrt{\frac{2}{\pi}}\frac{y}{y^2+x^2}\right)}_{P_y(x)} = \widehat{P_y}(\omega)$$

The function $P_y(x)$ is called the *Poisson kernel*.

Therefore

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{-y|\omega|} = \widehat{f}(\omega)\widehat{P_y}(\omega) = \widehat{f * P_y}(\omega).$$

Finally, we apply Fourier inversion to find that

$$\begin{aligned} u(x, y) &= (f * P_y)(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)P_y(x - s)\, ds \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{y^2 + (x - s)^2}\, ds, \end{aligned}$$

which is known as the *Poisson integral formula* for the solution to the Dirichlet problem on the upper half-plane.