Periodic functions and Fourier series

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Partial Differential Equations
Lecture 5

Goal: Given a function $f(x)$, write it as a linear combination of cosines and sines of increasing frequency, e.g.

$$\begin{aligned} f(x) &= a_0 + a_1\cos(x) + a_2\cos(2x) + \cdots + b_1\sin(x) + b_2\sin(2x) + \cdots \\ &= a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right). \end{aligned}$$

Important Questions:

1. Which f have such a *Fourier series expansion*?

   Difficult to answer completely. We will give sufficient conditions only.

2. Given f, how can we determine $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$?

   We will give explicit formulae. These involve the ideas of *inner product* and *orthogonality*.

Periodicity

Definition: A function $f(x)$ is T*-periodic* if

$$f(x+T) = f(x) \text{ for all } x \in \mathbb{R}.$$

Remarks:

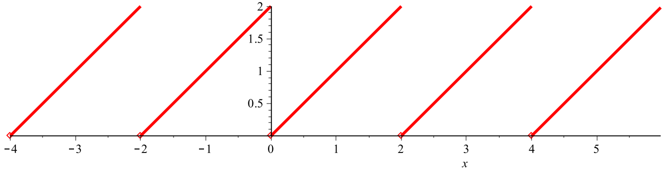
- If $f(x)$ is T-periodic, then $f(x+nT) = f(x)$ for any $n \in \mathbb{Z}$.
- The graph of a T-periodic function $f(x)$ repeats every T units along the x-axis.
- To give a formula for a T-periodic function, state that "$f(x) = \cdots$ for $x_0 \leq x < x_0 + T$" and then *either*:
    - $f(x+T) = f(x)$ for all x; OR
    - $f(x) = f\left(x - T\left\lfloor \frac{x - x_0}{T} \right\rfloor\right)$ for all x.

Examples

1. $\sin(x)$ and $\cos(x)$ are 2π-periodic.
2. $\tan(x)$ is π-periodic.
3. If $f(x)$ is T-periodic, then:
   - $f(x)$ is also nT-periodic for any $n \in \mathbb{Z}$.
   - $f(kx)$ is T/k-periodic.
4. For $n \in \mathbb{N}$, $\cos(nkx)$ and $\sin(nkx)$ are:
   - $2\pi/nk$-periodic.
   - *simultaneously* $2\pi/k$-periodic.
5. If $f(x)$ is T-periodic, then

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx \text{ for all } a.$$

6. The 2-periodic function with graph



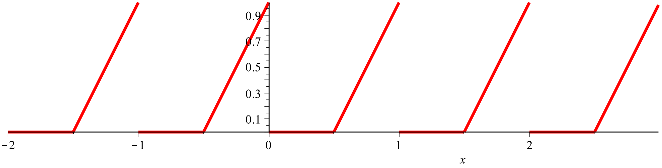
can be described by

$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 2, \\ f(x+2) & \text{for all } x, \end{cases}$$

or

$$f(x) = x - 2\left\lfloor \frac{x}{2} \right\rfloor.$$

7. The 1-periodic function with graph



can be described by

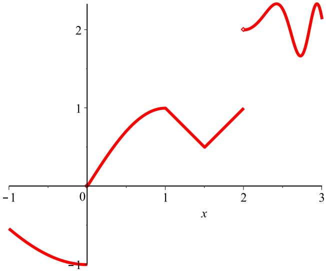
$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1/2, \\ 2x - 1 & \text{if } 1/2 < x \leq 1, \\ f(x+1) & \text{for all } x. \end{cases}$$

Piecewise smoothness

Definition: Given a function $f(x)$ we define

$$f(c+) = \lim_{x\to c^+} f(x) \text{ and } f(c-) = \lim_{x\to c^-} f(x).$$

Example: For the following function we have:



$$f(0+) = 0,$$
$$f(0-) = -1,$$
$$f(1+) = f(1) = f(1-) = 1,$$
$$f(2+) = 2,$$
$$f(2-) = 1.$$

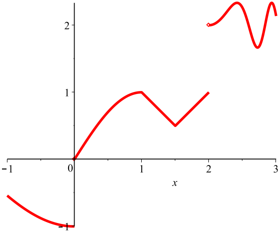
Remark: $f(x)$ is continuous at c iff $f(c) = f(c+) = f(c-)$.

Definition 1: We say that $f(x)$ is *piecewise continuous* if

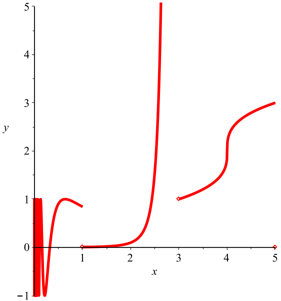
- f has only finitely many discontinuities in any interval, and
- $f(c+)$ and $f(c-)$ exist for all c in the domain of f.

Definition 2: We say that $f(x)$ is *piecewise smooth* if f *and* f' are both piecewise continuous.

Good:



Bad:



Remark: A piecewise smooth function *cannot* have: vertical asymptotes, vertical tangents, or "strange" discontinuities.

Which functions have Fourier series?

We noted earlier that the functions

$$\cos(nx),\ \sin(nx),\ (n \in \mathbb{N})$$

are all 2π-periodic. It follows that if

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

then $f(x)$ *must* also be 2π-periodic (linear combinations of T-periodic functions are T-periodic.)

However, 2π-periodicity alone *does not* guarantee that $f(x)$ has a Fourier series expansion.

But if we also require $f(x)$ to be piecewise smooth...

Existence of Fourier series

Theorem

If $f(x)$ is a piecewise smooth, 2π-periodic function, then there are (unique) Fourier coefficients $a_0, a_1, a_2, \ldots$ *and* $b_1, b_2, \ldots$ *so that*

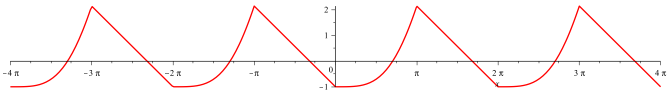
$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

for all x. This is called the Fourier series *of $f(x)$.*

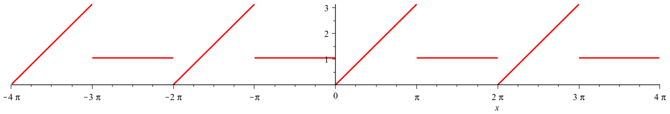
Remarks:

- If f is continuous at x, then $(f(x+) + f(x-))/2 = f(x)$. So f equals its Fourier series at "most points."
- If f is continuous everywhere, then f equals its Fourier series everywhere.

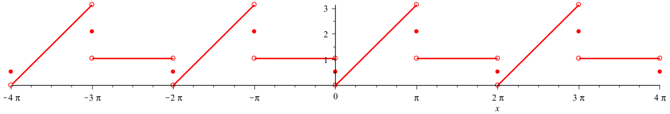
A continuous 2π-periodic function equals its Fourier series.



A discontinuous 2π-periodic piecewise smooth function...



...is *almost* its Fourier series.



Inner products and orthogonality in $\mathbb{R}^n$

Given vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, their *inner (or dot) product* is

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We say that $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Useful facts from linear algebra:

- For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle, \quad \langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a \langle \mathbf{x}, \mathbf{z} \rangle + b \langle \mathbf{y}, \mathbf{z} \rangle,$$
$$\langle \mathbf{x}, \mathbf{x} \rangle > 0 \text{ unless } \mathbf{x} = \mathbf{0}.$$

- A set of n orthogonal vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$ (an *orthogonal basis*).

Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ be orthogonal vectors in $\mathbb{R}^n$.

According to the facts above, given $\mathbf{x} \in \mathbb{R}^n$, there are (unique) *Fourier coefficients* $a_1, a_2, \ldots, a_n$ so that

$$\mathbf{x} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n.$$

We can use the inner product to help us compute these coefficients, e.g.

$$\begin{aligned} \langle \mathbf{b}_1, \mathbf{x} \rangle &= \langle \mathbf{b}_1, a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n \rangle \\ &= a_1 \langle \mathbf{b}_1, \mathbf{b}_1 \rangle + a_2 \langle \mathbf{b}_1, \mathbf{b}_2 \rangle + \cdots + a_n \langle \mathbf{b}_1, \mathbf{b}_n \rangle \\ &= a_1 \langle \mathbf{b}_1, \mathbf{b}_1 \rangle + 0 + \cdots + 0, \end{aligned}$$

which shows $a_1 = \langle \mathbf{b}_1, \mathbf{x} \rangle / \langle \mathbf{b}_1, \mathbf{b}_1 \rangle$.

In general we have the following result.

Theorem

If $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ *is an orthogonal basis of* $\mathbb{R}^n$ *and* $\mathbf{x} \in \mathbb{R}^n$*, then*

$$\mathbf{x} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n$$

where the Fourier coefficients of $\mathbf{x}$ *are given by*

$$a_i = \frac{\langle \mathbf{b}_i, \mathbf{x} \rangle}{\langle \mathbf{b}_i, \mathbf{b}_i \rangle} \quad (i = 1, 2, \ldots, n).$$

Example

Show that the vectors $\mathbf{b}_1 = (1, 0, 1)$*,* $\mathbf{b}_2 = (1, 1, -1)$ *and* $\mathbf{b}_3 = (-1, 2, 1)$ *form an orthogonal basis for* $\mathbb{R}^3$*, and express* $\mathbf{x} = (1, 2, 3)$ *in terms of this basis.*

It is easy to see that $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = \langle \mathbf{b}_1, \mathbf{b}_3 \rangle = \langle \mathbf{b}_2, \mathbf{b}_3 \rangle = 0$, e.g.

$$\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = 1 \cdot 1 + 0 \cdot 1 + 1 \cdot (-1) = 0.$$

The theorem tells us that the Fourier coefficients of $\mathbf{x} = (1, 2, 3)$ relative to this basis are

$$a_1 = \frac{\langle \mathbf{b}_1, \mathbf{x} \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} = \frac{1 \cdot 1 + 0 \cdot 2 + 1 \cdot 3}{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1} = 2,$$
$$a_2 = \frac{\langle \mathbf{b}_2, \mathbf{x} \rangle}{\langle \mathbf{b}_2, \mathbf{b}_2 \rangle} = \frac{1 \cdot 1 + 1 \cdot 2 + (-1) \cdot 3}{1 \cdot 1 + 1 \cdot 1 + (-1) \cdot (-1)} = 0,$$
$$a_3 = \frac{\langle \mathbf{b}_3, \mathbf{x} \rangle}{\langle \mathbf{b}_3, \mathbf{b}_3 \rangle} = \frac{(-1) \cdot 1 + 2 \cdot 2 + 1 \cdot 3}{(-1) \cdot (-1) + 2 \cdot 2 + 1 \cdot 1} = 1.$$

That is,

$$\mathbf{x} = 2\mathbf{b}_1 + \mathbf{b}_3.$$

Inner products of functions

Goal: Find an inner product of *functions* that will allow us to "extract" Fourier coefficients.

Definition: Given two functions $f(x)$ and $g(x)$, their *inner product (on the interval $[a, b]$)* is

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

Example: The inner product of x and x^2 on $[0, 1]$ is

$$\left\langle x, x^2\right\rangle = \int_0^1 x \cdot x^2\,dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}.$$

Remarks: If f, g, h are functions and c, d are constants, then:

- $\langle f, g \rangle = \langle g, f \rangle$;
- $$\begin{aligned} \langle cf + dg, h \rangle &= \int_a^b (cf(x) + dg(x))h(x)\,dx \\ &= c\int_a^b f(x)h(x)\,dx + d\int_a^b g(x)h(x)\,dx \\ &= c\,\langle f, h \rangle + d\,\langle g, h \rangle\,; \end{aligned}$$
- $\langle f, f \rangle = \displaystyle\int_a^b f(x)^2\,dx \geq 0$;
- $\langle f, f \rangle = 0$ iff $f \equiv 0$;
- we say f and g are *orthogonal on* $[a, b]$ if $\langle f, g \rangle = 0$.

Examples

1. The functions $1 - x^2$ and x are orthogonal on $[-1, 1]$ since

$$\langle 1 - x^2, x \rangle = \int_{-1}^{1} (1 - x^2)x \, dx = \left. \frac{x^2}{2} - \frac{x^4}{4} \right|_{-1}^{1} = 0.$$

2. The functions $\sin x$ and $\cos x$ are orthogonal on $[-\pi, \pi]$ since

$$\langle \sin x, \cos x \rangle = \int_{-\pi}^{\pi} \sin x \cos x \, dx = \left. \frac{\sin^2 x}{2} \right|_{-\pi}^{\pi} = 0.$$

3. More generally, for any $m, n \in \mathbb{N}_0$, the functions $\sin(mx)$ and $\cos(nx)$ are orthogonal on $[-\pi, \pi]$ since

$$\langle \sin(mx), \cos(nx) \rangle = \int_{-\pi}^{\pi} \underbrace{\sin(mx)\cos(nx)}_{\text{odd}} \, dx = 0.$$

4. For any $m, n \in \mathbb{N}_0$, consider $\cos(mx)$ and $\cos(nx)$ on $[-\pi, \pi]$.

$$\begin{aligned}
\langle \cos(mx), \cos(nx) \rangle &= \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \cos\left((m+n)x\right) + \cos\left((m-n)x\right)\,dx \\
(m \neq n) \qquad &= \frac{1}{2}\left(\frac{\sin\left((m+n)x\right)}{m+n} + \frac{\sin\left((m-n)x\right)}{m-n}\right)\Bigg|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

since $\sin(k\pi) = 0$ for $k \in \mathbb{Z}$. If $m = n$, then we have

$$\begin{aligned}
\langle \cos(mx), \cos(mx) \rangle &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(2mx) + 1\,dx \\
(m \neq 0) \qquad &= \frac{1}{2}\left(\frac{\sin(2mx)}{2m} + x\right)\Bigg|_{-\pi}^{\pi} = \pi.
\end{aligned}$$

Finally, if $m = n = 0$, then

$$\langle \cos(mx), \cos(mx) \rangle = \langle 1, 1 \rangle = \int_{-\pi}^{\pi} dx = 2\pi.$$

We conclude that on $[-\pi, \pi]$ one has

$$\langle \cos(mx), \cos(nx) \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n \neq 0, \\ 2\pi & \text{if } m = n = 0. \end{cases}$$

5. Likewise, on $[-\pi, \pi]$ one can show that for $m, n \in \mathbb{N}$

$$\langle \sin(mx), \sin(nx) \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

Conclusion

Relative to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx,$$

the functions occurring in every Fourier series, namely

$$1, \cos(x), \cos(2x), \cos(3x) \ldots, \sin(x), \sin(2x), \sin(3x), \ldots$$

form an orthogonal set.

Moral: We can use the inner product above to “extract” Fourier coefficients via integration!