More on Fourier Series

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Partial Differential Equations
Lecture 7

General Fourier series

If $f(x)$ is $2p$-periodic and piecewise smooth, then $\hat{f}(x) = f(px/\pi)$ has period $\frac{2p}{p/\pi} = 2\pi$, and is also piecewise smooth.

It follows that $\hat{f}(x)$ has a Fourier series:

$$\frac{\hat{f}(x+) + \hat{f}(x-)}{2} = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)).$$

Since $f(x) = \hat{f}(\pi x/p)$, we find that f also has a Fourier series:

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right).$$

We can use Euler's formulas to find a_n and b_n. For example

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{px}{\pi}\right)\,dx = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt,$$

where in the final equality we used the substitution $t = px/\pi$.

In the same way one can show that for $n \geq 1$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\left(\frac{n\pi t}{p}\right)\,dt,$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\left(\frac{n\pi t}{p}\right)\,dt.$$

Since t is simply a "dummy" variable of integration, we may replace it with x in each case.

Remarks on general Fourier series

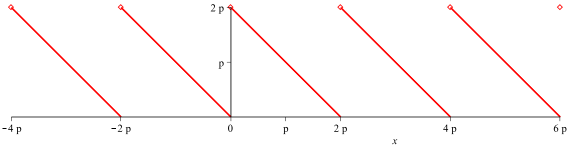
Everything we've done with 2π-periodic Fourier series continues to hold in this case, with p replacing π:

- We can compute general Fourier coefficients by integrating over any "convenient" interval of length $2p$.
- If p is left unspecified, then the formulae for a_n and b_n may involve p.
- If $f(x)$ is even, then $b_n = 0$ for all n.
- If $f(x)$ is odd, then $a_n = 0$ for all n.
- We still have the uniform convergence theorem and Wilbraham-Gibbs phenomenon.

Example

Find the Fourier series of the $2p$-periodic function that satisfies $f(x) = 2p - x$ for $0 \leq x < 2p$.

The graph of $f(x)$:



We will use Euler's formulas over the interval $[0, 2p]$ to simplify our calculations.

We have

$$a_0 = \frac{1}{2p}\int_0^{2p} 2p - x\,dx = \frac{1}{2p}\left(2px - \frac{x^2}{2}\bigg|_0^{2p}\right) = p$$

and for $n \geq 1$

$$\begin{aligned} a_n &= \frac{1}{p}\int_0^{2p}(2p-x)\cos\left(\frac{n\pi x}{p}\right)\,dx \\ &= \frac{1}{p}\left(\frac{p(2p-x)\sin(\frac{n\pi x}{p})}{n\pi} - \frac{p^2\cos(\frac{n\pi x}{p})}{n^2\pi^2}\Bigg|_0^{2p}\right) \\ &= \frac{1}{p}\left(-\frac{p^2\cos(2n\pi)}{n^2\pi^2} + \frac{p^2}{n^2\pi^2}\right) = 0, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{p}\int_0^{2p}(2p-x)\sin\left(\frac{n\pi x}{p}\right)\,dx \\ &= \frac{1}{p}\left(\frac{-p(2p-x)\cos(\frac{n\pi x}{p})}{n\pi} - \frac{p^2\sin(\frac{n\pi x}{p})}{n^2\pi^2}\bigg|_0^{2p}\right) \\ &= \frac{1}{p}\left(\frac{2p^2}{n\pi}\right) = \frac{2p}{n\pi}. \end{aligned}$$

So the Fourier series of f is

$$p + \sum_{n=1}^{\infty}\frac{2p}{n\pi}\sin\left(\frac{n\pi x}{p}\right) = p + \frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{p}\right).$$

Remark: This series is equal to $f(x)$ everywhere it is continuous.

Differentiating Fourier series

Term-by-term differentiation of a series can be a useful operation, *when it is valid*. The following result tells us when this is the case with Fourier series.

Theorem

Suppose f is 2π-periodic and piecewise smooth. If f' is also piecewise smooth, and f is continuous everywhere, then the Fourier series for f' can be obtained from that of f using term-by-term differentiation.

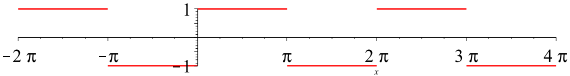
Remark: This can be proven by using integration by parts in the Euler formulas for the Fourier coefficients of f'.

Example

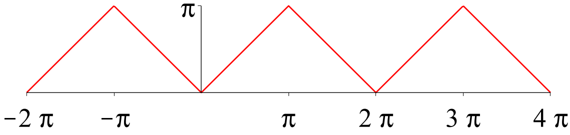
Use an existing series to find the Fourier series of the 2π-periodic function satisfying

$$f(x) = \begin{cases} -1 & \text{if } -\pi \le x < 0, \\ 1 & \text{if } 0 \le x < \pi. \end{cases}$$

The graph of $f(x)$ (a *square wave*)



shows that it is the derivative of the triangular wave.



Since the triangular wave is continuous everywhere, we can differentiate its Fourier series term-by-term to get the series for the square wave.

$$\frac{d}{dx}\left(\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)x)}{(2k+1)^2}\right) = -\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{-(2k+1)\sin((2k+1)x)}{(2k+1)^2}$$
$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{(2k+1)}.$$

Warning: The hypothesis that f is continuous is *extremely important*. For example, if we term-wise differentiate the Fourier series for the *discontinuous* square wave (above), we get

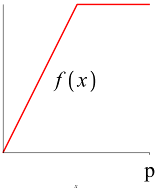
$$\frac{4}{\pi}\sum_{k=0}^{\infty}\cos((2k+1)x)$$

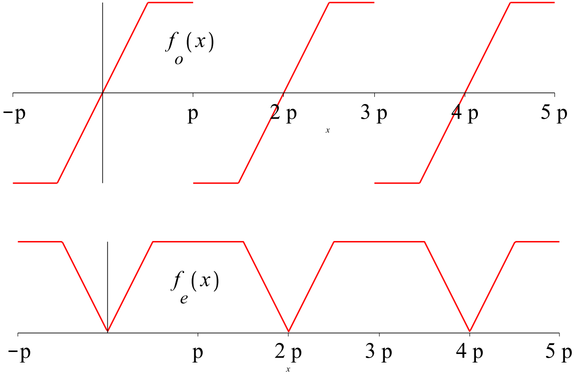
which converges (almost) *nowhere*!

Half-range expansions

Goal: Given a function $f(x)$ defined for $0 \le x \le p$, write $f(x)$ as a linear combination of sines and cosines.

Idea: Extend f to have period $2p$, and find the Fourier series of the resulting function.





Sine and cosine series

We set

$$f_o = \text{odd } 2p\text{-periodic extension of } f,$$
$$f_e = \text{even } 2p\text{-periodic extension of } f.$$

If we expand f_o as a Fourier series, it will involve only sines:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right).$$

This is the *sine series expansion* of f.

According to Euler's formula the Fourier coefficients are given by

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f_o(x)\sin\left(\frac{n\pi x}{p}\right)}_{\text{even}} dx = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right) dx.$$

If we expand f_e as a Fourier series, it will involve only cosines:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right).$$

This is the *cosine series expansion* of f.

This time Euler's formulas give

$$a_0 = \frac{1}{2p}\int_{-p}^{p} \underbrace{f_e(x)}_{\text{even}}\, dx = \frac{1}{p}\int_0^p f(x)\, dx,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f_e(x)\cos\left(\frac{n\pi x}{p}\right)}_{\text{even}}\, dx = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\, dx.$$

If f is piecewise smooth, both the sine and cosine series converge to the function $\dfrac{f(x+)+f(x-)}{2}$ (on the interval $[0,p]$).

Example

Find the sine and cosine series expansions of $f(x) = 3 - x$ on the interval $0 \leq x \leq 3$.

Taking $p = 3$ in our work above, the coefficients of the sine series are given by

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 (3-x)\sin\left(\frac{n\pi x}{3}\right)\,dx \\ &= \frac{2}{3}\left(\frac{-3(3-x)}{n\pi}\cos\left(\frac{n\pi x}{3}\right) - \frac{9}{n^2\pi^2}\sin\left(\frac{n\pi x}{3}\right)\Big|_0^3\right) \\ &= \frac{2}{3}\cdot\frac{9}{n\pi}\cos(0) = \frac{6}{n\pi}. \end{aligned}$$

So, the sine series is

$$\frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{3}\right).$$

The cosine series coefficients are

$$a_0 = \frac{1}{3}\int_0^3 3 - x\,dx = \frac{1}{3}\left(3x - \frac{x^2}{2}\bigg|_0^3\right) = \frac{3}{2}$$

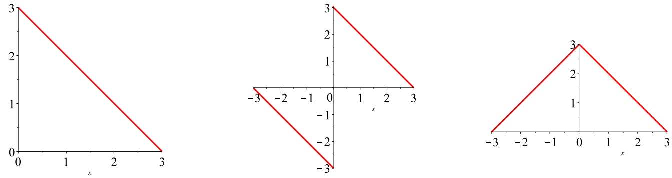
and for $n \geq 1$

$$\begin{aligned} a_n &= \frac{2}{3}\int_0^3 (3-x)\cos\left(\frac{n\pi x}{3}\right)\,dx \\ &= \frac{2}{3}\left(\frac{3(3-x)}{n\pi}\sin\left(\frac{n\pi x}{3}\right) - \frac{9}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)\bigg|_0^3\right) \\ &= \frac{2}{3}\left(-\frac{9}{n^2\pi^2}\cos(n\pi) + \frac{9}{n^2\pi^2}\right) = \begin{cases} \dfrac{12}{n^2\pi^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Since we can omit the terms with even n, we write $n = 2k + 1$ ($k \geq 0$) and obtain the cosine series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) = \frac{3}{2} + \frac{12}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left(\frac{(2k+1)\pi x}{3}\right).$$

Here are the graphs of f, f_o and f_e (over one period):



Consequently, the sine series equals $f(x)$ for $0 < x \leq 3$, and the cosine series equals $f(x)$ for $0 \leq x \leq 3$.