Linear PDEs and the Principle of Superposition

Ryan C. Daileda

Trinity University

Partial Differential Equations
Lecture 9

Linear differential operators

Definition: A *linear differential operator* (in the variables $x_1, x_2, \ldots x_n$) is a sum of terms of the form

$$A(x_1, x_2, \ldots, x_n) \frac{\partial^{a_1 + a_2 + \cdots + a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}},$$

where each $a_i \geq 0$.

Examples: The following are linear differential operators.

1. The Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

2. $W = c^2 \nabla^2 - \dfrac{\partial^2}{\partial t^2}$

3. $H = c^2\nabla^2 - \dfrac{\partial}{\partial t}$

4. $T = \dfrac{\partial}{\partial t} - v_1\dfrac{\partial}{\partial x_1} - v_2\dfrac{\partial}{\partial x_2} - \cdots - v_n\dfrac{\partial}{\partial x_n} = \dfrac{\partial}{\partial t} - \mathbf{v}\cdot\nabla$

5. The general first order linear operator (in two variables):

$$D_1 = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y} + C(x,y)$$

6. The general second order linear operator (in two variables):

$$\begin{aligned} D_2 = A(x,y)\frac{\partial^2}{\partial x^2} + 2B(x,y)\frac{\partial^2}{\partial x\partial y} + C(x,y)\frac{\partial^2}{\partial y^2} \\ + D(x,y)\frac{\partial}{\partial x} + E(x,y)\frac{\partial}{\partial y} + F(x,y) \end{aligned}$$

Theorem

If D is a linear differential operator (in the variables $x_1, x_2, \cdots x_n$), u_1 and u_2 are functions (in the same variables), and c_1 and c_2 are constants, then

$$D(c_1 u_1 + c_2 u_2) = c_1 D u_1 + c_2 D u_2.$$

Remarks:

- This follows immediately from the fact that each partial derivative making up D has this property, e.g.

$$\frac{\partial^3}{\partial x_1^2 \partial x_2}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial^3 u_1}{\partial x_1^2 \partial x_2} + c_2 \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}.$$

- This property extends (in the obvious way) to any number of functions and constants.

Linear PDEs

Definition: A *linear PDE* (in the variables $x_1, x_2, \cdots, x_n$) has the form

$$Du = f \tag{1}$$

where:

- D is a linear differential operator (in $x_1, x_2, \cdots, x_n$),
- f is a function (of $x_1, x_2, \cdots, x_n$).

We say that (1) is *homogeneous* if $f \equiv 0$.

Examples

The following are linear PDEs.

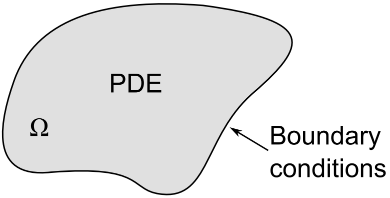
1. The Laplace equation: $\nabla^2 u = 0$ (homogeneous)
2. The wave equation: $c^2\nabla^2 u - \dfrac{\partial^2 u}{\partial t^2} = 0$ (homogeneous)
3. The heat equation: $c^2\nabla^2 u - \dfrac{\partial u}{\partial t} = 0$ (homogeneous)
4. The Poisson equation: $\nabla^2 u = f(x_1, x_2, \ldots, x_n)$ (inhomogeneous if $f \not\equiv 0$)
5. The advection equation: $\dfrac{\partial u}{\partial t} + \kappa\dfrac{\partial u}{\partial x} + ru = k(x,t)$ (inhomogeneous if $k \not\equiv 0$)
6. The telegraph equation: $\dfrac{\partial^2 u}{\partial t^2} + 2B\dfrac{\partial u}{\partial t} - c^2\dfrac{\partial^2 u}{\partial x^2} + Au = 0$ (homogeneous)

Boundary value problems

A *boundary value problem* (BVP) consists of:

- a domain $\Omega \subseteq \mathbb{R}^n$,
- a PDE (in n independent variables) to be solved in the interior of Ω,
- a collection of *boundary conditions* (BCs) to be satisfied on the boundary of Ω.

The data for a BVP:



Linear boundary conditions

Definition: Let $\Omega \subseteq \mathbb{R}^n$ be the domain of a BVP and let A be a subset of the boundary of Ω.

We say that a BC on A is *linear* if it has the form

$$\delta u\big|_A = f\big|_A \qquad (2)$$

where:

- δ is a linear differential operator (in $x_1, x_2, \cdots, x_n$),
- f is a function (of $x_1, x_2, \cdots, x_n$).

(The notation $\cdot\big|_A$ means "restricted to A.") We say that (2) is *homogeneous* if $f \equiv 0$.

Examples

The following are linear BCs.

1. *Dirichlet conditions*: $u|_A = f|_A$, such as
$$u(x,0) = f(x) \text{ for } 0 < x < L, \text{ or } u(L,t) = 0 \text{ for } t > 0$$

2. *Neumann conditions*: $\left.\frac{\partial u}{\partial \mathbf{n}}\right|_A = f|_A$, where $\frac{\partial u}{\partial \mathbf{n}}$ is the directional derivative perpendicular to A, such as
$$u_t(x,0) = g(x) \text{ for } 0 < x < L, \text{ or } u_x(0,t) = 0 \text{ for } t > 0$$

3. *Robin conditions*: $\left.u + a\frac{\partial u}{\partial \mathbf{n}}\right|_A = f|_A$, such as
$$u(L,t) + u_x(L,t) = 0 \text{ for } t > 0$$

The principle of superposition

Theorem

Let D and δ be linear differential operators (in the variables $x_1, x_2, \ldots, x_n$), let f_1 and f_2 be functions (in the same variables), and let c_1 and c_2 be constants.

- *If u_1 solves the linear PDE $Du = f_1$ and u_2 solves $Du = f_2$, then $u = c_1 u_1 + c_2 u_2$ solves $Du = c_1 f_1 + c_2 f_2$. In particular, if u_1 and u_2 both solve the same* homogeneous *linear PDE, so does $u = c_1 u_1 + c_2 u_2$.*
- *If u_1 satisfies the linear BC $\delta u|_A = f_1|_A$ and u_2 satisfies $\delta u|_A = f_2|_A$, then $u = c_1 u_1 + c_2 u_2$ satisfies $\delta u|_A = c_1 f_1 + c_2 f_2|_A$. In particular, if u_1 and u_2 both satisfy the same* homogeneous *linear BC, so does $u = c_1 u_1 + c_2 u_2$.*

Remarks on the superposition principle

- It is an easy consequence of the linearity of D, δ, e.g. if $Du_1 = f_1$ and $Du_2 = f_2$, then

$$D(c_1 u_1 + c_2 u_2) = c_1 Du_1 + c_2 Du_2 = c_1 f_1 + c_2 f_2.$$

- It extends (in the obvious way) to any number of functions and constants.

- It implies that linear combinations of functions that satisfy homogeneous linear PDEs/BCs satisfy *the same equations.*

Non-example

Warning: The principle of superposition can *easily* fail for nonlinear PDEs or boundary conditions.

Consider the nonlinear PDE

$$u_x + u^2 u_y = 0.$$

One solution of this PDE is

$$u_1(x, y) = \frac{-1 + \sqrt{1 + 4xy}}{2x}.$$

However, the function $u = cu_1$ *does not* solve the same PDE unless $c = 0, \pm 1$.

Superposition and separation of variables

Consider a linear BVP consisting of the following data:

(A) A *homogeneous* linear PDE on a region $\Omega \subseteq \mathbb{R}^n$;

(B) A (finite) list of *homogeneous* linear BCs on (part of) $\partial\Omega$;

(C) A (finite) list of *inhomogeneous* linear BCs on (part of) $\partial\Omega$.

Roughly speaking, to solve such a problem one:

1. Finds all "separated" solutions to (A) and (B).
   - This amounts to solving a collection of linear ODE BVPs linked by separation constants.
   - Superposition guarantees *any linear combination* of separated solutions also solves (A) and (B).
2. Determines the specific linear combination of separated solutions that solves (C).

Remarks on separation of variables

- When separated solutions involve sines and cosines, finding the solutions to inhomogeneous BCs utilize Fourier series/half-range expansions.
- More generally, one must make use of "Fourier like" series involving other families of orthogonal functions (e.g. Sturm-Liouville theory).
- When there are *no* homogeneous BCs, or "too many" inhomogeneous BCs, one can "homogenize" parts of the problem and then superimpose these partial results to get the complete solution.
- Depending on the shape of the domain in question, successful separation of variables may require change of coordinates.