MODERN ALGEBRA
SPRING 2023

ASSIGNMENT 11
DUE APRIL 5

Exercise 1. Let p be a prime number and let A be an abelian p-group. Show that every subgroup and every quotient of A is also an abelian p-group. [*Suggestion.* If $A' \leq A$, the natural map $\pi : A \to A/A'$ is a homomorphism. Now see Exercise 5.1.4.]

Exercise 2. Let p be a prime number and define

$$\mathbb{Z}_{(p)} = \left\{ \frac{n}{p^m} \,\middle|\, n \in \mathbb{Z}, m \in \mathbb{N}_0 \right\}.$$

a. Show that $\mathbb{Z}_{(p)}$ is a subgroup of $(\mathbb{Q}, +)$ containing $\mathbb{Z}$.

b. Show that $Z(p^\infty) := \mathbb{Z}_{(p)}/\mathbb{Z}$ is an infinite p-group.

Exercise 3. Let A and B be abelian groups, and let $f : A \to B$ be a homomorphism. Suppose there is a homomorphism $g : B \to A$ so that $f \circ g = 1_B$. Prove that $A = \ker f \oplus \operatorname{im} g$. [*Suggestion:* Given $a \in A$, write $a = (a - g(f(a))) + g(f(a))$.]