Modern Algebra
Spring 2023

Assignment 12.1
Due April 12

Exercise 1. Let C be an (additive) cyclic group of order n, generated by a, and suppose $n = de$ for some $d, e \in \mathbb{N}$. Prove that

$$C[d] = \langle ea \rangle .$$

Exercise 2. Let G be a group and let $\mathcal{H}$ be a *directed* set of subgroups of G: for all $H, K \in \mathcal{H}$ there exists a $J \in \mathcal{H}$ so that $H \cup K \leq J$. Prove that

$$L = \bigcup_{H \in \mathcal{H}} H$$

is a subgroup of G.

Exercise 3. An (additive) abelian group A is said to be a *torsion* group if every element of A has finite order. Given a prime p define

$$A(p) = \bigcup_{e=1}^{\infty} A[p^e],$$

which consists of all the elements of A whose order is some power of p.

a. Use the preceding exercise to show that $A(p) \leq A$.

b. Show that

$$A = \sum_p A(p),$$

where the (internal) sum runs over all primes p.[1] [*Suggestion.* If $0 \neq a \in A$ and $n = |a|$, then $a \in A[n]$. Express $A[n]$ as the internal direct sum of prime-power-torsion subgroups $A[p^e]$, and use the fact that $A[p^e] \leq A(p)$.]

c. Show that the internal sum $\sum_p A(p)$ is direct. [*Suggestion.* Suppose $a_i \in A(p_i)$ for some primes $p_1, p_2, \ldots, p_r$ and that $a_1 + \cdots + a_r = 0$. Then for each i, $a_i \in A[p_i^{e_i}]$ for some $e_i \geq 1$. But the sum $A[p_1^{e_1}] + \cdots + A[p_r^{e_r}]$ is direct (why?), so that $a_i = 0$ for all i.]

[1] The *sum* of a (possibly infinite) collection $\mathcal{B}$ of subgroups of A is defined to be the subgroup of A generated by $S = \bigcup_{B \in \mathcal{B}} B$. It consists of all *finite* sums of elements in S.