Modern Algebra — Assignment 13.1
Spring 2023 — Due April 19

Exercise 1. Let A be an additive abelian group and let $A_1, A_2, \ldots, A_n \leq A$. Let

$$A' = A_1 + A_2 + \cdots + A_n = \{a_1 + a_2 + \cdots + a_n \,|\, a_i \in A_i \text{ for all } i\}.$$

Define $f : A_1 \times A_2 \times \cdots \times A_n \to A$ by

$$f(a_1, a_2, \ldots, a_n) = a_1 + a_2 + \cdots + a_n.$$

a. Prove that f is a homomorphism and that $\operatorname{im} f = A'$. Conclude that A' is a subgroup of A.

b. Prove that the following conditions are equivalent.

- **i.** $f : A_1 \times A_2 \times \cdots \times A_n \to A'$ is an isomorphism.
- **ii.** If $a_i, b_i \in A_i$ for all i and

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n,$$

  then $a_i = b_i$ for all i.
- **iii.** For all i, we have

$$A_i \cap \sum_{j \neq i} A_j = \{0\}.$$

When the (equivalent) conditions of part **b** hold, we say that the sum of the A_i is *direct*, and we write

$$A_1 + A_2 + \cdots + A_n = A_1 \oplus A_2 \oplus \cdots \oplus A_n.$$

Exercise 2. Lang, II.4.4

Exercise 3. Lang, II.4.28