Exercise 1. Let p be an odd prime. It can be shown that the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ is always cyclic. Use this to prove that for any $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ one has

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}},$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. This result is known as *Euler's criterion.*

Exercise 2. Let χ, ψ be characters of a finite abelian group A. Define $(\chi\psi) : A \to \mathbb{C}^\times$ by $(\chi\psi)(a) = \chi(a)\psi(a)$ for all $a \in A$. Show that $\chi\psi$ is a character of A.

Exercise 3. Let A be a finite abelian group. The *trivial character* of A is defined by $\chi_0(a) = 1$ for all $a \in A$. If χ is any character of $|A|$, prove that

$$\sum_{a \in A} \chi(a) = \begin{cases} |A| & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

[*Suggestion.* Let S denote the sum in question. Show that for any $a \in A$ one has $\chi(a)S = S$. This implies $S(\chi(a) - 1) = 0$. Use this to conclude that $S = 0$ if $\chi \neq \chi_0$.]