Modern Algebra
Spring 2023

Assignment 14.2
Due April 26

Exercise 1. Let A be a finite abelian group. For $\chi, \psi \in \widehat{A}$, define their *inner product* to be

$$\langle \chi, \psi \rangle = \frac{1}{|A|} \sum_{a \in A} \chi(a) \overline{\psi(a)},$$

where for a complex number $z = x + iy$ we define $\overline{z} = x - iy$ (the *complex conjugate*). Use exercise 13.3.3 to prove the *orthogonality relations*

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

[*Suggestion:* Show that $|\psi(a)| = 1$ and hence $\overline{\psi(a)} = \psi(a)^{-1} = \psi^{-1}(a)$.]

Exercise 2. Let A be a finite abelian group of order n. Because A is self-dual, this implies $|\widehat{A}| = n$. Enumerate A and $\widehat{A}$ in some fashion:

$$A = \{a_1, a_2, \langle, a_n\}, \quad \widehat{A} = \{\chi_1, \chi_2, \ldots, \chi_n\}.$$

The *character table* of A is the $n \times n$ matrix

$$M = \frac{1}{\sqrt{n}} (\chi_i(a_j)),$$

whose rows (up to the factor of $1/\sqrt{n}$) give all the values of a fixed character. Use the orthogonality relations of the preceding exercise to prove that

$$MM^* = I,$$

where $M^* = \overline{M}^t$.

Exercise 3. Continue using the notation of Exercise 2. In linear algebra one usually proves that if P and Q are square matrices with the same dimensions and $PQ = I$, then $P = Q^{-1}$. In other words, if P is a left inverse for Q, then it is automatically a two-sided inverse. Exercise 2 then implies that $M^*M = I$. Use this to prove that for any $a, b \in A$ one has the *dual orthogonality relations*

$$\frac{1}{|A|} \sum_{\chi \in \widehat{A}} \chi(a) \overline{\chi}(b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

This dual relationship is an essential ingredient in the proof of Dirichlet's theorem on primes in arithmetic progressions.