Modern Algebra
Spring 2023

Assignment 15.2
Due April 28

Exercise 1. Let G be a finite group and let $H \leq G$. We have seen that G acts on the coset space G/H by left translation:

$$g(xH) = (gx)H$$

for all $g \in G$ and $xH \in G/H$. Because this is an action, the map $\lambda : G \to \mathrm{Perm}(G/H)$ given by $g \mapsto \lambda_g$, where $\lambda_g(xH) = g(xH) = (gx)H$, is a homomorphism.

a. If $K = \ker \lambda$, show that $K \leq H$.

b. Conclude that $[G : H]$ divides $[G : K]$.

Exercise 2. Continuing the preceding exercise, suppose p is the smallest prime dividing $|G|$, and that $[G : H] = p$. Then $\mathrm{Perm}(G/H) \cong S_p$ and the First Isomorphism Theorem applied to λ yields an embedding $G/K \hookrightarrow S_p$.

a. Explain why $[G : K]$ divides $p!$.

b. Use the preceding exercise to write $[G : K] = mp$ for some $m \in \mathbb{N}$. Use part **a** to show that m divides $(p-1)!$.

c. If a prime q divides m, show that $q < p$ and q divides $|G|$. Conclude that $m = 1$, so that $[G : K] = p$.

d. Show that $[H : K] = 1$. Conclude that $H \lhd G$.

Remark. Exercise 2 proves that if G is a finite group, $H \leq G$ and $[G : H]$ is the smallest prime dividing $|G|$, then $H \lhd G$. We proved a weaker version of this result in an earlier exercise, in the case that $|G|$ is even and $[G : H] = 2$, by directly comparing the left and right coset spaces G/H and $H \setminus G$.