MODERN ALGEBRA
SPRING 2023

ASSIGNMENT 3.1
DUE FEBRUARY 1

Exercise 1. Let G be a group and suppose that $x \in G$ has finite order $n \in \mathbb{N}$.

a. Prove that for any $m \in \mathbb{Z}$, $x^m = e$ if and only if n divides m. Conclude that

$$\{m \in \mathbb{Z} \,|\, x^m = e\} = n\mathbb{Z},$$

where $n\mathbb{Z} = \{nk \,|\, k \in \mathbb{Z}\}$. [*Suggestion.* Use the Division Algorithm to divide m by n. Use the minimality of n to argue that the remainder must be 0.]

b. Prove that the sequence $\{x^k\}_{k\in\mathbb{Z}}$ is periodic with minimal period n. This is the reason that some authors use the word *period* instead of *order*.

Exercise 2. Let G be a finite group. Prove that if G has even order, then G contains an element with order 2. [*Suggestion.* Count the elements of G by pairing them with their inverses.]

Exercise 3. Let G be a group and let $a, b \in G$. Denote the order of a by $|a|$.[1] Prove the following assertions.

a. $|a| = |a^{-1}|$

b. $|ab| = |ba|$

c. $|a| = |bab^{-1}|$

Exercise 4. Let G be a finite abelian group, written multiplicatively. Let $a \in G$.

a. Explain why the value of $\displaystyle\prod_{x\in G} x$ is independent of the particular ordering of G used to compute it.

b. Explain why $\displaystyle\prod_{x\in G} x = \prod_{x\in G} (ax) = a^{|G|} \prod_{x\in G} x$.

c. Use part **b** and Exercise **1a** to conclude that $|a|$ divides $|G|$.

[1] Do not assume that $|a|$ is necessarily finite.