Exercise 1. Define $f : D_n \to \{\pm 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rotation,} \\ -1 & \text{otherwise.} \end{cases}$$

Prove that f is a homomorphism. What is its kernel?

Exercise 2. Lang, Exercise II.3.11.

Exercise 3. Let A be an abelian group and let $n \in \mathbb{N}$. We say that n is an *exponent* for A provided $na = 0$ for all $a \in A$.[1] Given $m \in \mathbb{N}$, recall that the m-torsion subgroup of A is

$$A[m] = \{a \in A \,|\, ma = 0\}.$$

So n is an exponent for A if and only if $A = A[n]$. Assume that this is the case in **a**–**c** below.

- **a.** Suppose $n = rs$ for some $r, s \in \mathbb{N}$. Let $E_r : A \to A$ be the homomorphism defined by $E_r(a) = ra$. Show that $\operatorname{im} E_r < A\,[s]$.
- **b.** Assuming that we also have $\gcd(r, s) = 1$, prove that $\operatorname{im} E_r = A\,[s]$. [*Suggestion.* Let $a \in A[s]$ and use Bézout's lemma to write $ur + vs = 1$ with $u, v \in \mathbb{Z}$. Then consider the equation $a = 1a$.]
- **c.** Prove that the map $f : A \to A[r] \oplus A[s]$ given by $f = E_s \oplus E_r$ is an isomorphism. [*Suggestion.* Use the previous suggestion to show $\ker f$ is trivial. Given $(b, c) \in A[r] \oplus A[s]$, show that $f(uc + vb) = (b, c)$.]

Exercise 4. Let A be an (additive) abelian group and $B, C < A$. Define $f : B \times C \to A$ by $f(b, c) = b + c$.

- **a.** Prove that f is a homomorphism.
- **b.** Use part **a** to prove that $B + C = \{b + c \,|\, b \in B, c \in C\}$ is a subgroup of A.
- **c.** Compute $\ker f$.

[1] The terminology is borrowed from the equivalent multiplicative formulation $a^n = e$ and is entirely standard, if not a little confusing.