Exercise 1. Let G be a group and let S be a set. Suppose that $f : G \to S$ is a bijection. We can use f to turn S into a group in a natural way. Given $x, y \in S$, we use f^{-1} to pull them back to G, apply the binary operation in G, and then push the result forward to S. That is, we define

$$x * y = f(f^{-1}(x) \cdot f^{-1}(y)).$$

Prove that $(S, *)$ is a group and that $f : (G, \cdot) \to (S, *)$ is an isomorphism.

Exercise 2. Let $(G, \cdot)$ be a multiplicative group. Given $a \in G$, recall that the left translation $\lambda_a : G \to G$ given by $\lambda_a(x) = ax$ is a bijection. In general λ_a is not an isomorphism. However, if we let $*$ denote the binary operation on G defined by taking $S = G$ and $f = \lambda_a$ in the preceding exercise, then $\lambda_a : (G, \cdot) \to (G, *)$ *is* an isomorphism. Show that $*$ agrees with the operation introduced in exercise 1.2.3.

Remark. Exercise 2 shows that every group G is *homogeneous.* For any $a \in G$ we can use λ_a to translate the original group structure on G to another group structure on G that uses a as the identity, and λ_a provides an isomorphism between the two. So G "looks the same" after left translation by any of its elements. This means that the set theoretic nature of G is irrelevant to any group structure imposed upon G. Put another way, if you know that G is a group, but you've forgetten its binary operation, there is no intrinsic way to recover it.

Exercise 3. If G is a multiplicative group and $H, K < G$, let

$$HK = \{ab \,|\, a \in H, b \in K\}.$$

This is the multiplicative analogue of the sum of two subgroups of an abelian group.

- **a.** Give an example to show that if G is nonabelian, then HK is not necessarily a subgroup of G.
- **b.** Suppose that $xHx^{-1} \subseteq H$ for all $x \in G$. Show that $HK = KH$ and that $HK < G$.