MODERN ALGEBRA
SPRING 2023

ASSIGNMENT 6.1
DUE FEBRUARY 22

Exercise 1. Let $n \in \mathbb{N}$. Recall that for $a, b \in \mathbb{Z}$ we say that a is *congruent to b modulo n*, denoted $a \equiv b \pmod{n}$, provided $n | a - b$. If $R_n(x)$ denotes the remainder when an integer x is divided by n using the Division Algorithm, show that $a \equiv b \pmod{n}$ if and only if $R_n(a) = R_n(b)$.

Exercise 2. Let $\ell, m, n \in \mathbb{Z}$ with $\ell | n$ and $m | n$. Let $d = \gcd\left(\frac{n}{\ell}, \frac{n}{m}\right)$ and set $A = \langle \ell \rangle + \langle m \rangle$ in $\mathbb{Z}_n$.

- **a.** Show that $A = \langle \ell \rangle \oplus \langle m \rangle$ if and only if $d = 1$.
- **b.** Show that $\mathbb{Z}_n = \langle \ell \rangle \oplus \langle m \rangle$ if and only if $d = 1$ and $n = \ell m$.

Exercise 3. Let G be a group and $S \subseteq G$. Given $x, y \in G$, define $x \sim y$ by the condition $xy^{-1} \in S$. Prove that if $\sim$ is an equivalence relation on G, then $S < G$.