Modern Algebra
Spring 2023

Assignment 6.2
Due February 22

Exercise 1. Let $n \in \mathbb{N}$. Compute $[\mathbb{Z} : n\mathbb{Z}]$. [*Suggestion.* Remember that the cosets of $n\mathbb{Z}$ in $\mathbb{Z}$ are precisely the congruence classes modulo n].

Exercise 2. Compute $[\mathbb{R}^\times : \mathbb{R}^+]$.

Exercise 3. Show that every coset of $\mathbb{R}^+$ in $\mathbb{C}^\times$ contains a unique element of absolute value equal to 1. Conversely, show that every coset of $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ in $\mathbb{C}^\times$ contains a unique positive real number. In both cases, describe the cosets geometrically.

Exercise 4. Prove that $[\mathbb{R} : \mathbb{Q}]$ is infinite. [*Suggestion:* Argue by contradiction.]

Exercise 5. Let G be a group of order pqr, where p, q and r are distinct primes. If $H, K < G$ satisfy $|H| = qp$ and $|K| = qr$, prove that $|H \cap K| = q$. [*Suggestion:* Observe that K has more elements than H has (left) cosets, then use the pigeonhole principle.]