Modern Algebra — Assignment 7
Spring 2023 — Due March 1

Definition: Given a group G and a subgroup H of G, we say that H is *normal* in G (denoted $H \triangleleft G$) if and only if

$$x^{-1}Hx \subseteq H \quad \text{for all } x \in G. \tag{N1}$$

Exercise 1. Let G be a group and $H < G$. Prove that if $[G : H] = 2$, then $H \triangleleft G$ [*Suggestion:* If $x \in G$ and $x \notin H$, explain why the cosets xH and Hx must coincide.]

Exercise 2. Prove that $\mathbb{Q}$ has no proper subgroups of finite index. [*Suggestion:* If $H < \mathbb{Q}$ has index n, then $\mathbb{Q}/H$ is a finite group of order n. Now notice that for any $r \in \mathbb{Q}$ one has $r + H = n \cdot \left(\frac{r}{n}\right) + H = n(\frac{r}{n} + H)$.]

Exercise 3. Let G, H be groups, $f : G \to H$ a homomorphism.

a. If $K \triangleleft H$, prove that $f^{-1}(K) \triangleleft G$.

b. If $K \triangleleft G$ and f is surjective, prove that $f(K) \triangleleft H$.

Exercise 4. Let G be a group, $H < G$ and $N \triangleleft G$.

a. Prove that $HN < G$.

b. Prove that $H \cap N \triangleleft H$.

c. Prove that if $H \triangleleft G$ and $H \cap N$ is trivial, then $hn = nh$ for all $h \in H$ and $n \in N$.