Modern Algebra
Spring 2023

Assignment 8.1
Due March 8

Exercise 1. Let G be a group and let $H < G$. Given $a \in G$, define

$$\Lambda_a : G/H \to G/H$$

by $\Lambda_a(xH) = (ax)H$.

- **a.** Show that Λ_a is well-defined. That is, if $x, y \in G$ and $xH = yH$, show that $(ax)H = (ay)H$.
- **b.** Prove that $\Lambda_a \in \mathrm{Perm}(G/H)$.
- **c.** Prove that the map $\Lambda : G \to \mathrm{Perm}(G/H)$, defined by $a \mapsto \Lambda_a$ for $a \in G$, is a homomorphism.

Exercise 2. Let G be a group and $H < G$. The *normalizer of H in G* is

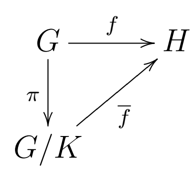
$$N_G(H) = \{x \in G \mid xHx^{-1} = H\}.$$

- **a.** Prove that $N_G(H)$ is a subgroup of G containing H, and that H is normal in $N_G(H)$.
- **b.** Prove that the set $\{xHx^{-1} \mid x \in G\}$ of conjugates of H is in one to one correspondence with the left cosets of $N_G(H)$ in G.

Exercise 3. Let $f : G \to H$ be a homomorphism of groups and set $K = \ker f$. Let $\overline{f} : G/K \to H$ be the induced map given by $\overline{f}(aK) = f(a)$, and let $\pi : G \to G/K$ be the canonical epimorphism. Suppose that $\varphi : G/K \to H$ is a homomorphism satisfying the conclusion of the First Isomorphism Theorem, namely

$$f = \varphi \circ \pi.$$

Show that $\varphi = \overline{f}$. This proves that $\overline{f} : G/K \to H$ is the unique homomorphism so that the diagram



commutes. [*Suggestion.* This is nearly trivial. Given $aK \in G/K$, use the fact that $aK = \pi(a)$ to show that $\varphi(aK) = \overline{f}(aK)$.]