**Exercise 1.** Let $G$ be a group and $H \triangleleft G$. Use the natural epimorphism $\pi : G \to G/H$ to prove the following.

**a.** If $G$ is abelian, then $G/H$ is abelian.

**b.** If $G$ is cyclic, then $G/H$ is cyclic.

**Exercise 2.** Let $p$ be a prime number and define

$$R_p = \left\{ \frac{r}{s} \in \mathbb{Q} \,\middle|\, p \nmid s \right\}.$$

**a.** Prove that $pR_p < R_p < \mathbb{Q}$.

**b.** Let $f : \mathbb{Z} \to R_p/pR_p$ be the homomorphism obtained by composing the inclusion $\mathbb{Z} \hookrightarrow R_p$ with the natural epimorphism $\pi : R_p \to R_p/pR_p$. Show that $f$ is surjective. [*Suggestion.* Given $r/s \in R_p$, the condition $p \nmid s$ implies that $s \in \mathbb{Z}_p^\times$. So we may choose $t \in \mathbb{Z}$ so that $st \equiv 1 \pmod{p}$. Show that $f(rt) = \frac{r}{s} + pR_p$.]

**c.** Show that $\ker f = p\mathbb{Z}$ and use the First Isomorphism Theorem to conclude that $\mathbb{Z}/p\mathbb{Z} \cong R_p/pR_p$.

**Exercise 3.** A *fractional linear transformation* is a nonconstant rational function of the form $\ell(x) = \frac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R}$.

**a.** Compute $\ell'(x)$ and conclude that $\ell(x)$ is nonconstant if and only if $ad - bc \neq 0$.

**b.** Let $\mathcal{L}(\mathbb{R})$ denote the set of fractional linear transformations. Prove that $\mathcal{L}(\mathbb{R})$ is a group under composition of functions.

**c.** Show that the map $f : \mathrm{GL}_2(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ax+b}{cx+d}$$

is a surjective homomorphism.

**d.** Show that $\ker f = Z(\mathrm{GL}_2(\mathbb{R}))$ and conclude that

$$\mathrm{GL}_2(\mathbb{R})/Z(\mathrm{GL}_2(\mathbb{R})) \cong \mathcal{L}(\mathbb{R}).$$