Row Reduction and Echelon Forms

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Linear Algebra

Recall

In order to efficiently solve the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

we first introduced the *augmented matrix*

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) = (A \mid \mathbf{b}).$$

We then implemented Gaussian elimination (*row reduction*) by performing *elementary row operations* on $(A \mid \mathbf{b})$ until we could simply recognize the solution(s) of the system.

Elementary Row Operations

- (Replacement) Add a multiple of one row to another row.
- (Scaling) Multiply all entries in a row by a *nonzero* constant.
- (Interchange/Swap) Interchange two rows.

Question. How do we know when to stop row reducing?

Answer. When we reach an *echelon form*.

Row Echelon Form

We will say that a row (or column) of a matrix is *nonzero* if it has at least one nonzero entry.

The leftmost entry in a nonzero row will be called its *leading entry*.

Definition (Row Echelon Form)

A matrix M is said to be in *row echelon form* (REF) iff:

1. All nonzero rows are above all rows of zeros.
2. The leading entry of any row is to the right of the leading entry above it.
3. All entries directly below a leading entry are zero.

Example

The following are (row) echelon forms. The symbol ■ denotes a *nonzero* entry, while $*$ denotes an arbitrary value.

$$\begin{pmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \blacksquare & * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{pmatrix}$$

Reduced Row Echelon Form

Definition (Reduced Row Echelon Form)

Suppose M is a matrix in row echelon form. We say that M is in *reduced row echelon form* (RREF) iff:

4. Every leading entry is equal to 1.
5. Each leading entry is the only nonzero entry in its column.

Here are the RREFs of the preceding examples.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & * & * & 0 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}$$

Row Reduction and Echelon Forms

We will soon give an algorithm for row reducing *any* matrix into an echelon form.

Using scaling and replacement operations, any echelon form is easily brought into *reduced* echelon form.

Depending on the operations used, different echelon forms may be obtained from the same matrix. However:

Theorem 1 (Uniqueness of Reduced Echelon Forms)

Every matrix can be row reduced to exactly one **reduced** *row echelon form.*

Pivots

Before we introduce the row reduction algorithm, we need some terminology.

Definition

The *pivot positions* of a matrix A are the locations of the leading entries in any echelon form for A. A *pivot row/column* is any row/column containing a pivot position.

For example, since

$$A = \begin{pmatrix} \boxed{1} & 2 & 4 & 8 \\ 2 & 4 & \boxed{6} & 8 \\ 3 & 6 & 9 & \boxed{12} \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the pivot positions of A are located in the boxes.

Row Reduction Algorithm

Step 1. Choose the leftmost nonzero column and use an interchange operation (if necessary) to put a nonzero entry at its top. This is a pivot position.

Step 2. Use replacement operations to make every entry below the pivot equal to zero.

Step 3. Cover the row containing the pivot and repeat Steps 1-3 on the submatrix that remains.

This will bring your matrix into row echelon form. To reduce it:

Step 4. Moving from right to left, scale each leading entry to 1 (if necessary) and use replacements to make all entries above it equal to 0.

Remarks

- This algorithm proves that every matrix has a reduced echelon form. It does *not* prove, however, that it is unique. We will take this for granted.
- Because the RREF of a matrix A is unique, the pivot positions are the same for *every* echelon form (reduced or otherwise) of A.
- A judicious choice of the pivot entry in Step 1 can sometimes be helpful.
- As long as we only use elementary row operations, strict adherence to the RREF algorithm is unnecessary. Different sequences of steps can sometimes be more efficient.

Example

Example 1

Find the RREF of

$$\begin{pmatrix} 3 & 9 & -4 & -2 & 3 \\ 3 & 9 & -5 & 6 & 20 \\ -1 & -3 & 2 & 1 & -1 \\ 1 & 3 & -1 & 2 & 6 \end{pmatrix}.$$

Solution. Because the leftmost column is nonzero, it is a pivot column. For convenience we swap rows to put the 1 in the pivot:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 3 & 9 & -5 & 6 & 20 \\ -1 & -3 & 2 & 1 & -1 \\ 3 & 9 & -4 & -2 & 3 \end{pmatrix}.$$

Now use replacements to eliminate the entries below the pivot:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & -1 & -8 & -15 \end{pmatrix}.$$

If we block out the row containing the pivot, the leftmost nonzero column is now the third, and the pivot position is shown below:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 0 & 0 & \boxed{-2} & 0 & 2 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & -1 & -8 & -15 \end{pmatrix}.$$

To make lower replacements easier, the most efficient thing to do is scale row 2 by -1/2 first.

We then have

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & -1 & -8 & -15 \end{pmatrix}.$$

Now use replacements to eliminate the entries below the second pivot:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -8 & -16 \end{pmatrix}.$$

Blocking out the pivot rows, the leftmost nonzero column is now the fourth. The pivot is in the $(3, 4)$-entry.

Again to save us some trouble later, we scale the fourth row by 1/3 before proceeding:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & -8 & -16 \end{pmatrix}.$$

Now use replacement to eliminate everything below the current pivot:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Blocking out the pivot rows leaves only zeros, so we have arrived at an echelon form.

The leading entries are all 1, so no scaling is needed. We simply need two replacements to eliminate the entries above the pivots:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 2 & 6 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-2R_3+R_1\to R_1} \begin{pmatrix} \boxed{1} & 3 & -1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

And finally:

$$\begin{pmatrix} \boxed{1} & 3 & -1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2+R_1\to R_1} \begin{pmatrix} \boxed{1} & 3 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced row echelon form. □

Remarks.

- Many computer algebra systems (Maple, Matlab, Mathematica, etc.) include routines for computing reduced row echelon forms.
- Although understanding the row reduction algorithm is important, we will frequently omit the steps involved in computing the RREF of a matrix.

Echelon Forms and Linear Systems

We motivate the general situation with an example.

Example 2

Solve the system

$$\begin{aligned} 3x_1 + 9x_2 - 4x_3 - 2x_4 &= 3, \\ 3x_2 + 9x_2 - 5x_3 + 6x_4 &= 20, \\ -x_1 - 3x_2 + 2x_3 + x_4 &= -1, \\ x_1 + 3x_2 - x_3 + 2x_4 &= 6. \end{aligned}$$

Solution. The augmented matrix $(A \mid \mathbf{b})$ of this system is just the matrix of the preceding example.

We have just seen that row reduction leads to the RREF

$$(A \mid \mathbf{b}) \to \left(\begin{array}{cccc|c} \boxed{1} & 3 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

which corresponds to the equivalent system

$$\begin{array}{ccccc} x_1 & +3x_2 & & & = & 1 \\ & & x_3 & & = & -1 \\ & & & x_4 & = & 2 \end{array}$$

(the last equation is 0=0, which we will omit).

Notice that *any* choice of x_2 easily leads to a solution of the system.

We therefore have the *parametric solution*

$$\boxed{\begin{aligned} x_1 &= 1 - 3x_2, \\ x_3 &= -1, \\ x_4 &= 2, \end{aligned}}$$

in which x_2 is a *free variable*. □

In general, given a linear system, the variables corresponding to pivot columns of the augmented matrix are called *basic variables*. The variables in non-pivot columns are called *free variables*.

Because of the structure of the RREF, it is easy to solve for the basic variables in terms of the free variables, and any choice of values for the free variables then gives a solution of the system.

Example

Example 3

Solve the system

$$\begin{aligned} 5x_1 - x_2 + 5x_3 + 9x_4 - 2x_5 &= -3, \\ 3x_1 \qquad + 6x_3 + 9x_4 - x_5 &= 4, \\ 2x_1 \qquad + 4x_3 + 6x_4 - x_5 &= 0. \end{aligned}$$

Solution. We have the augmented matrix

$$\left(\begin{array}{ccccc|c} 5 & -1 & 5 & 9 & -2 & -3 \\ 3 & 0 & 6 & 9 & -1 & 4 \\ 2 & 0 & 4 & 6 & -1 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 1 & 5 & 6 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{array}\right).$$

This corresponds to the system

$$\begin{array}{ccccccc} x_1 & & +2x_3 & +3x_4 & & = & 4, \\ & x_2 & +5x_3 & +6x_4 & & = & 7, \\ & & & & x_5 & = & 8. \end{array}$$

The free variables are x_3 and x_4 so that the solutions have the parametric form

$$\boxed{\begin{array}{lcl} x_1 & = & 4 - 2x_3 - 3x_4, \\ x_2 & = & 7 - 5x_3 - 6x_4, \\ x_3 & \text{is} & \text{free}, \\ x_4 & \text{is} & \text{free}, \\ x_5 & = & 8. \end{array}}$$

$\square$

Existence, Uniqueness and Echelon Forms

Using echelon forms, we can give complete answers to the questions of existence and uniqueness of solutions to linear systems.

Our results show that a linear system is guaranteed to be *consistent* (have solutions) iff the RREF of its augmented matrix *does not* yield an equation of the form $0 = b$, where b is *nonzero*.

That is, the RREF *cannot* contain a row of the form

$$\begin{pmatrix} 0 & 0 & 0 \cdots & 0 & b \end{pmatrix}, \text{ with } b \neq 0.$$

Equivalently, there is *not* a pivot in the last column of the augmented matrix.

Furthermore, if the system is consistent, it will have a unique solution iff there are *no* free variables. Otherwise it has infinitely many solutions.

So we will have a unique solution iff the augmented matrix has a pivot in *every* column except the last.

To summarize:

Theorem 2

A linear system is consistent iff its augmented matrix does not *have a pivot in the last column. In this case, the solution is unique iff there is a pivot in every column* but the last. *Otherwise there are infinitely many solutions.*

Example

Example 4

Show that a consistent system of 4 linear equations in 5 unknowns must have infinitely many solutions.

Solution. Because there cannot be more than one pivot per row, the 4×6 augmented matrix of the system can have at most 4 pivots.

Thus we *cannot* have a pivot for each of the 5 variables. So the solution *cannot* be unique.

Since we are told the system is consistent, according to Theorem 2 the only remaining possibility is that there are infinitely many solutions.

□

Remarks

We will say that an $m \times n$ linear system is:

- *underdetermined* if $m < n$ (i.e. there are more variables than equations; augmented matrix is “long”)

- *overdetermined* if $m > n$ (i.e. there are more equations than variables; augmented matrix is “tall”).

The result of Example 4 is easily generalized: a consistent underdetermined system must have infinitely many solutions!

So in order for a linear system to have a *unique* solution, there must be *at least as many equations as unknowns* (however, this does not guarantee the system is actually consistent).