Rank

Let A be an $m \times n$ matrix with

$$\text{rank}(A) = \# \text{ pivots in } A = r.$$

Then we must have

$$r \leq m \quad \text{and} \quad r \leq n,$$

since there can't be more than one pivot per row or column.

The extreme cases are of special interest.

Full Column Rank ($r = n$ and $n \le m$)

1. All columns are pivot columns (columns are independent). No free variables in $A\mathbf{x} = \mathbf{b}$.

2. $\mathbf{N}(A)$ contains only the zero vector: $\mathbf{N}(A) = \{\mathbf{0}\}$.

3. *If* $A\mathbf{x} = \mathbf{b}$ has a solution, it is unique.

Full Row Rank ($r = m$ and $m \leq n$)

1. All rows are pivot rows and R_0 has no zero rows (so $R = R_0$).
2. $A\mathbf{x} = \mathbf{b}$ has a solution for *any* $\mathbf{b} \in \mathbb{R}^m$.
3. The column space is all of $\mathbb{R}^m$: $\mathbf{C}(A) = \mathbb{R}^m$.
4. If $m < n$, then $\mathbf{N}(A) \neq \{\mathbf{0}\}$ and $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Rank and $A\mathbf{x} = \mathbf{b}$

Dimensions/Rank	Features	Solutions to $A\mathbf{x} = \mathbf{b}$
$r = m = n$	Square and invertible	Exactly 1 solution
$r = m$ and $r < n$	Short and wide	∞ solutions
$r < m$ and $r = n$	Tall and thin	0 or 1 solutions
$r < m$ and $r < n$	Not full rank	0 or ∞ solutions