Example 1 Revisited

We put the system on the left and the corresponding augmented matrix on the right. Denote the ith equation by E_i and the ith row of the augmented matrix by R_i.

$$\begin{array}{rrcr} x_1 & +5\,x_2 & = & 7 \\ -2\,x_1 & -7\,x_2 & = & -5 \end{array} \quad \rightsquigarrow \quad \left(\begin{array}{rr|r} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right)$$

$$[2E_1 + E_2 \to E_2] \qquad\qquad [2R_1 + R_2 \to R_2]$$

$$\begin{array}{rrcr} x_1 & +5x_2 & = & 7 \\ & 3x_2 & = & 9 \end{array} \qquad\qquad \left(\begin{array}{rr|r} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right)$$

$$[\tfrac{1}{3}E_2 \to E_2] \qquad\qquad [\tfrac{1}{3}R_2 \to R_2]$$

$$\begin{array}{rcrcr} x_1 & +5x_2 & = & 7 \\ & x_2 & = & 3 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right)$$

$$[-5E_2 + E_1 \to E_2] \qquad [-5R_2 + R_1 \to R_1]$$

$$\begin{array}{rcrcr} x_1 & & = & -8 \\ & x_2 & = & 3 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right)$$

So the only solution is $\boxed{x_1 = -8 \text{ and } x_2 = 3.}$

Example 2

Solve the linear system. As above, denote the ith equation by E_i and the ith row of the augmented matrix by R_i.

$$\begin{array}{rrrl} & x_2 & +5x_3 & = -4 \\ x_1 & +4x_2 & +3x_3 & = -2 \\ 2x_1 & +7x_2 & -x_3 & = 1 \end{array} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & -1 & 1 \end{array}\right)$$

$$[E_1 \leftrightarrow E_2] \qquad [R_1 \leftrightarrow R_2]$$

$$\begin{array}{rrrl} x_1 & +4x_2 & +3x_3 & = -2 \\ & x_2 & +5x_3 & = -4 \\ 2x_1 & +7x_2 & -x_3 & = 1 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & -1 & 1 \end{array}\right)$$

$$[-2E_1 + E_3 \to E_3] \qquad [-2R_1 + R_3 \to R_3]$$

$$\begin{array}{rrrl} x_1 & +4x_2 & +3x_3 & = -2 \\ & x_2 & +5x_3 & = -4 \\ & -x_2 & -7x_3 & = 5 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -7 & 5 \end{array}\right)$$

$$[E_2 + E_3 \to E_3] \qquad [R_2 + R_3 \to R_3]$$

$$\begin{array}{rrrl} x_1 & +4x_2 & +3x_3 & = -2 \\ & x_2 & +5x_3 & = -4 \\ & & -2x_3 & = 1 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

$$[-\tfrac{1}{2}E_3 \to E_3] \qquad [-\tfrac{1}{2}R_3 \to R_3]$$

$$\begin{array}{rrrl} x_1 & +4x_2 & +3x_3 & = -2 \\ & x_2 & +5x_3 & = -4 \\ & & x_3 & = -\frac{1}{2} \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

$$[-5E_3 + E_2 \to E_2] \qquad [-5R_3 + R_2 \to R_2]$$

$$[-3E_3 + E_1 \to E_1] \qquad [-3R_3 + R_1 \to R_1]$$

$$\begin{array}{rrrl} x_1 & +4x_2 & & = -\frac{1}{2} \\ & x_2 & & = -\frac{3}{2} \\ & & x_3 & = -\frac{1}{2} \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 4 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

$$[-4E_2 + E_1 \to E_1] \qquad [-4R_2 + R_1 \to R_1]$$

$$
\begin{aligned}
x_1 \qquad\qquad &= \tfrac{11}{2} \\
x_2 \qquad &= -\tfrac{3}{2} \\
x_3 &= -\tfrac{1}{2}
\end{aligned}
\qquad\qquad
\left(\begin{array}{ccc|c}
1 & 0 & 0 & \frac{11}{2} \\
0 & 1 & 0 & -\frac{3}{2} \\
0 & 0 & 1 & -\frac{1}{2}
\end{array}\right)
$$

So the only solution is $\boxed{x_1 = \frac{11}{2},\ x_2 = -\frac{3}{2} \text{ and } x_3 = -\frac{1}{2}.}$

Example 3

Use augmented matrices to solve the linear system.

$$\begin{array}{ccccc} x_1 & -2x_2 & -6x_3 & = 12 \\ 2x_1 & +4x_2 & +12x_3 & = -17 \\ x_1 & -4x_2 & -12x_3 & = 22 \end{array} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{array}\right)$$

$$\begin{array}{c} [-2R_1 + R_2 \to R_2] \\ [-R_1 + R_3 \to R_3] \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & 8 & 24 & -41 \\ 0 & -2 & -6 & 10 \end{array}\right) [R_2 \leftrightarrow R_3] \left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 8 & 24 & -41 \end{array}\right)$$

Finally $4R_2 + R_3 \to R_3$ produces

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{array}\right) \qquad \rightsquigarrow \qquad \begin{array}{rrrl} x_1 & -2x_2 & -6x_3 & = 12 \\ & -2x_2 & -6x_3 & = 10 \\ & & 0 & = -1 \end{array}$$

This system clearly has **no solutions**.

Example 4

Use augmented matrices to solve the (nearly identical) system.

$$\begin{array}{rrrl} x_1 & -2x_2 & -6x_3 & = 12 \\ 2x_1 & +4x_2 & +12x_3 & = -16 \\ x_1 & -4x_2 & -12x_3 & = 22 \end{array} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -16 \\ 1 & -4 & -12 & 22 \end{array}\right)$$

This time the exact same row operations yield

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right) [-R_2 + R_1 \to R_1] \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

After the scaling $[-\frac{1}{2}R_2 \to R_2]$ we finally end up with

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{array}{rrrl} x_1 & & & = 2 \\ & x_2 & +3x_3 & = -5 \\ & & 0 & = 0 \end{array}$$

This tells us that $\boxed{x_1 = 2 \text{ and } x_2 = -3x_3 - 5 \text{ for } \textit{any} \text{ choice of } x_3.}$

That is, there are *infinitely many* solutions.