MODERN ALGEBRA
SPRING 2025

ASSIGNMENT 1.2
DUE JANUARY 22

Exercise 1. Given $n \in \mathbb{N}$ and $a \in \mathbb{Z}$, let $R_n(a)$ denote the remainder when a is divided by n using the Division Algorithm.

a. Prove that for any $a, b \in \mathbb{Z}$, $R_n(a) = R_n(b)$ if and only if $a - b$ is divisible by n in $\mathbb{Z}$, i.e. there is a $c \in \mathbb{Z}$ so that $a - b = cn$.

b. Prove that for any $a, b \in \mathbb{Z}$ one has $R_n(a+b) = R_n(R_n(a)+b)$ and $R_n(ab) = R_n(R_n(a)b)$.

c. Recall that for $a, b \in \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ we defined $a \oplus b = R_n(a+b)$ and $a \otimes b = R_n(ab)$. Show that $\oplus$ and $\otimes$ are both associative binary operations on $\mathbb{Z}_n$. [*Suggestion.* Given $a, b, c \in \mathbb{Z}_n$, consider $R_n(a+b+c)$ and $R_n(abc)$, and use part **b**.]

Exercise 2. Let

$$G = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\},$$

where $i^2 = -1$.

a. Show that G is closed under matrix multiplication (that is, $AB \in G$ for all $A, B \in G$).

b. Prove that G is a group under matrix multiplication. Is it abelian?

Exercise 3. Let G be a group and $e_0 \in G$. Given $x, y \in G$, define a new binary operation $*$ on G by

$$x * y = x e_0^{-1} y.$$

Prove that G is a group under $*$.