MODERN ALGEBRA
SPRING 2025

ASSIGNMENT 11.2
DUE APRIL 16

Exercise 1. Let p be a prime number. If G is a group, $N \lhd G$, and N and G/N are both p-groups, prove that G is a p-group.

Exercise 2. Let p be a prime number and let A be an abelian p-group. Show that every subgroup and every quotient of A is also an abelian p-group. [*Suggestion.* If $A' < A$, apply Exercise 5.1.5 to the natural epimorphism $\pi : A \to A/A'$.]

Exercise 3. Let p be a prime number and define

$$\mathbb{Z}[p^{-1}] = \left\{ \frac{n}{p^m} \,\middle|\, n \in \mathbb{Z}, m \in \mathbb{N}_0 \right\}.$$

a. Show that $\mathbb{Z}[p^{-1}]$ is a subgroup of $(\mathbb{Q}, +)$ containing $\mathbb{Z}$.

b. Show that the *Prüfer group*

$$\mathbb{Z}(p^\infty) := \mathbb{Z}[p^{-1}]/\mathbb{Z}$$

is an infinite p-group.

c. Show that every subgroup of $\mathbb{Z}(p^\infty)$ has the form $\left(\frac{1}{p^m}\mathbb{Z}\right)/\mathbb{Z}$ for some $m \in \mathbb{N}_0$, which is cyclic of order p^m.[†]

[†]This means that $\mathbb{Z}(p^\infty)$ contains (a unique copy of) $\mathbb{Z}/p^m\mathbb{Z}$ for all $m \in \mathbb{N}_0$.