Modern Algebra
Spring 2025

Assignment 12.2
Due April 23

Exercise 1. Let A be an additive abelian group with subgroups $B_1, B_2, \ldots, B_r$.

a. Show that the function
$$f : \prod_{i=1}^{r} B_i \to A$$
given by $f(b_1, b_2, \ldots, b_r) = b_1 + b_2 + \cdots + b_r$ is a homomorphism with image $B_1 + B_2 + \cdots + B_r$.

b. Show that f is an isomorphism onto $B_1 + B_2 + \cdots + B_r$ if and only if for all i and all $b_i \in B_i$, the condition $b_1 + b_2 + \cdots + b_r = 0$ implies $b_1 = b_2 = \cdots = b_r = 0$.

When f is an isomorphism we say that the internal sum $B_1 + B_2 + \cdots + B_r$ is *direct*, which we denote by writing $B_1 \oplus B_2 \oplus \cdots \oplus B_r$.

Exercise 2. Let p be a prime number and let $r \in \mathbb{N}$. If $C(p^r)$ denotes an additive cyclic group of order p^r, show that
$$pC(p^r) = C(p^{r-1})$$
[*Suggestion.* Consider the endomorphism $C(p^r) \to C(p^r)$ given by $x \mapsto px$.]

Exercise 3. Let A be an additive finite abelian p-group. If A is nontrivial, prove that pA is a proper subgroup of A.

Exercise 4. Let p be a prime number. Classify the finite abelian p-groups of order p^6 (there are 11 isomorphism classes).