Modern Algebra
Spring 2025

Assignment 13.2
Due April 30

Exercise 1. Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, the complex *upper half-plane*. Show that for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{C}$, the rule

$$Az = \frac{az + b}{cz + d}$$

defines an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$.

Exercise 2. Let G be a group and let S be a set with a G-action. For $x \in G$, define $T_x : S \to S$ by $T_x(s) = xs$ for all $s \in S$. Show that $T_x \in \mathrm{Perm}(S)$ and that the map $\alpha : G \to \mathrm{Perm}(S)$ given by $\alpha(x) = T_x$ is a homomorphism. What is $\ker \alpha$?

Exercise 3. Prove that the converse of Exercise 2 is also true. That is, show that if $\alpha : G \to \mathrm{Perm}(S)$ is a homomorphism, then the rule $x \cdot s = (\alpha(x))(s)$ defines a G-action on S.

Exercise 4. Let S be a G-set and suppose that $H \triangleleft G$ satisfies $hs = s$ for all $h \in H$ and $s \in S$. Prove that the rule

$$(xH) \cdot s = xs \quad \text{for } x \in G, s \in S$$

is a well-defined G/H-action on S. [*Suggestion.* This can be proven directly from the definitions, or more "cleanly" as follows: let $\alpha : G \to \mathrm{Perm}(S)$ be the homomorphism of Exercises 2 and 3, compute $\alpha(H)$, and apply the preliminary version of the First Isomorphism Theorem.]