Modern Algebra — Spring 2025

Assignment 3.3 — Due February 12

Exercise 1. For $z = a + bi \in \mathbb{C}$, where $a, b \in \mathbb{R}$ and $i^2 = -1$, recall that we defined $|z| = \sqrt{a^2 + b^2}$. Let $\overline{z} = a - bi$ denote the *complex conjugate* of z.

- **a.** Show that for all $z, w \in \mathbb{C}$ one has $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.
- **b.** Show that for all $z \in \mathbb{C}$ one has $|z|^2 = z\overline{z}$.
- **c.** Use parts **a** and **b** to show that $|zw| = |z|\,|w|$ for all $z, w \in \mathbb{C}$.

Exercise 2. For $n \in \mathbb{N}$, let $\boldsymbol{\mu}_n = \{z \in \mathbb{C}^\times \mid z^n = 1\}$. The elements of $\boldsymbol{\mu}_n$ are called the nth *roots of unity.*

- **a.** Prove that $\boldsymbol{\mu}_n < S^1$.
- **b.** *Euler's Formula* states that
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
for any $\theta \in \mathbb{R}$. Show that $z \in S^1$ if and only if $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.
- **c.** The addition formulae for sine and cosine imply that
$$e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$$
for any $\theta, \phi \in \mathbb{R}$. Use this fact and part **b** to show that
$$z \in \boldsymbol{\mu}_n \iff z = e^{\frac{2k\pi i}{n}}$$
for some $k \in \mathbb{Z}$. Conclude that $|\boldsymbol{\mu}_n| = n$.

Exercise 3. Prove that $Z(\mathrm{GL}_2(\mathbb{R})) = \{aI \mid a \in \mathbb{R}^\times\}$. [*Suggestion:* First show that $\{aI \mid a \in \mathbb{R}^\times\}$ is a subgroup of the center. To prove the opposite inclusion, choose any $A \in Z(\mathrm{GL}_2(\mathbb{R}))$ and commute it with the matrices $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$.]

Exercise 4. Let
$$\mathrm{O}_2(\mathbb{R}) = \left\{A \in \mathrm{GL}_2(\mathbb{R}) \mid AA^t = I\right\},$$
where A^t denotes the transpose of A (reflection of its entries across the diagonal). Show that $\mathrm{O}_2(\mathbb{R}) < \mathrm{GL}_2(\mathbb{R})$.

Exercise 5. Let S be a nonempty set, let $a \in S$ and define

$$\mathcal{F}_a = \{\sigma \in \mathrm{Perm}(S) \mid \sigma(a) = a\},$$

the permutations of S that have a as a *fixed point*. Prove that $\mathcal{F}_a < \mathrm{Perm}(S)$.