Modern Algebra
Spring 2025

Assignment 4.1
Due February 19

Exercise 1. Let $m, n \in \mathbb{N}_0$. Prove that $m\mathbb{Z} = n\mathbb{Z}$ if and only if $m = n$.

Exercise 2. Let $a, b \in \mathbb{Z}$. Use the classification of the subgroups of $\mathbb{Z}$ to prove that

$$\langle a, b \rangle = a\mathbb{Z} + b\mathbb{Z} = \gcd(a, b)\,\mathbb{Z}$$

and that

$$a\mathbb{Z} \cap b\mathbb{Z} = \operatorname{lcm}(a, b)\mathbb{Z}.$$

For $n \in \mathbb{Z}$ we are using the alternate (number theoretic) notation $n\mathbb{Z} = \langle n \rangle$ for the subgroup of $\mathbb{Z}$ generated by n.

Exercise 3. A group G is called *finitely generated* if there exist $x_1, x_2, \ldots, x_n \in G$ so that $G = \langle x_1, x_2, \ldots, x_n \rangle$.

- **a.** Prove that $\mathbb{Z}^n = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ summands}}$ can be generated by n elements, and no fewer (this requires a little bit of linear algebra).
- **b.** Prove that $\mathbb{Q}$ is *not* finitely generated. [*Suggestion:* Argue by contradiction.]

Exercise 4. Lang, Exercise II.1.5. [*Warning*: This is *not* simply the explicit form of $\langle S \rangle$ stated in class.]