Modern Algebra
Spring 2025

Assignment 6.1
Due March 5

Exercise 1. Let G be a group and $S \subseteq G$. Given $x, y \in G$, define $x \sim y$ by the condition $xy^{-1} \in S$. Prove that if $\sim$ is an equivalence relation on G, then $S < G$.

Exercise 2. Prove that the index is "multiplicative in towers." That is, if $K < H < G$ are groups, and $[G : H]$ and $[H : K]$ are both finite, then $[G : K]$ is finite and

$$[G : K] = [G : H]\,[H : K].$$

You *may not* assume G is finite.

Exercise 3. We know that $\mathbb{R}^+ < \mathbb{C}^\times$ and $S^1 < \mathbb{C}^\times$. Use *polar coordinates* in $\mathbb{C}^\times$ (see the following page) below.

- **a.** Geometrically describe the elements of $\mathbb{C}^\times/\mathbb{R}^+$ (as subsets of $\mathbb{C} = \mathbb{R}^2$). [*Remark.* $\mathbb{C}^\times$ is abelian, so left and right cosets of $\mathbb{R}^+$ are the same.]
- **b.** Geometrically describe the elements of $\mathbb{C}^\times/S^1$.

Exercise 4. Let G be a group and let H and K be subgroups of finite index in G. Prove that $H \cap K$ also has finite index in G.

Polar coordinates in $\mathbb{C}^\times$

Given any $z \in \mathbb{C}^\times$, the multiplicativity of the absolute value implies $z/|z| \in S^1$. Thus, by Exercise 3.3.2, $z/|z| = e^{i\theta}$ for some $\theta \in \mathbb{R}$ and so

$$z = |z| \left(\frac{z}{|z|} \right) = |z| e^{i\theta} = re^{i\theta},$$

where $r = |z| > 0$. This is called the *polar form* of z, and (r, θ) are the *polar coordinates* of z (under the identification $x + iy \mapsto (x, y)$ of $\mathbb{C}$ with $\mathbb{R}^2$, these are the usual polar coordinates; check this). The absolute value $r = |z|$ is frequently called the *modulus* of z, and θ is called "the" *argument* of z. Clearly r is unique to z, but θ can be replaced by any member of the coset $\theta + 2\pi\mathbb{Z}$ (of the subgroup $2\pi\mathbb{Z}$ in $\mathbb{R}$). So θ should really be thought of as a member of $\mathbb{R}/2\pi\mathbb{Z}$ (we'll have a precise interpretation of this soon).

The polar expression of $z \in \mathbb{C}^\times$ is much more useful than its Cartesian expression, because it interacts very nicely with multiplication, and actually provides a geometric interpretation for multiplication of complex numbers (addition is just addition in $\mathbb{R}^2$, which is no more than vector addition, or translation). If $z = re^{i\theta}$ and $w = \rho e^{i\phi}$ with $r, \rho > 0$, then

$$zw = (r\rho) e^{i(\theta + \phi)},$$

which shows that *to multiply nonzero complex numbers, multiply their moduli and add their arguments.* Put another way, multiplication of $\mathbb{C}^\times$ by $z = re^{i\theta}$ is dilation by $r = |z|$ followed by a counterclockwise rotation (about the origin) of θ radians.