Modern Algebra
Spring 2025

Assignment 6.2
Due March 5

Exercise 1. Describe the elements of $R^\times/\mathbb{R}^+$. Use this to compute $[\mathbb{R}^\times : \mathbb{R}^+]$.

Exercise 2. If G is a multiplicative abelian group and $n \in \mathbb{N}$, we know that $g \mapsto g^n$ is an endomorphism of G, whose image is clearly

$$P_n(G) = \{g^n \mid g \in G\},$$

the nth powers in G. This implies immediately that $P_n(G) < G$.

a. Show that $[\mathbb{C}^\times : P_n(\mathbb{C}^\times)] = 1$.

a. Show that $[\mathbb{R}^\times : P_n(\mathbb{R}^\times)] = \frac{3+(-1)^n}{2}$.

b. Show that $[\mathbb{Q}^\times : P_n(\mathbb{Q}^\times)]$ is infinite for $n \geq 2$ (it actually equals $|\mathbb{Z}|$).

Exercise 3. Show that $[\mathbb{R} : \mathbb{Q}]$ is infinite (it actually equals $|\mathbb{R}|$) [*Suggestion.* Argue by contradiction.]

Exercise 4. Let G be a group of order pqr, where p, q and r are distinct primes. If $H, K < G$ satisfy $|H| = qp$ and $|K| = qr$, prove that $|H \cap K| = q$. [*Suggestion.* Observe that K has more elements than H has (left) cosets, then use the pigeonhole principle.]