Modern Algebra
Spring 2025

Assignment 6.3
Due March 5

Exercise 1. Let G be a group. Prove that $Z(G) \lhd G$.

Exercise 2. Let $f : G \to H$ be a group homomorphism.

- **a.** Show that $[a, b] \in f^{-1}([H, H])$ for all $a, b \in G$. Conclude that $[G, G] \subseteq f^{-1}([H, H])$, and hence $f([G, G]) \subseteq [H, H]$.
- **b.** Use part **a** to show that if $f : G \to H$ is an isomorphism, then $f([G, G]) = [H, H]$.
- **c.** According to exercise 4.3.2, for any $g \in G$ the conjugation map $c_g : G \to G$ given by $c_g(x) = gxg^{-1}$ is an automorphism of G. Use this and part **b** to show that $[G, G] \lhd G$.

Exercise 3. Let G be a group and suppose $H \lhd G$, so that the binary operation $(xH)(yH) = (xy)H$ on G/H is well-defined. Prove that this operation is commutative if and only if $[G, G] \subseteq H$.

Exercise 4. Let $n \in \mathbb{N}$, $n \geq 2$. Let

$$\Gamma(n) = \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv I \pmod{n}\}$$

and

$$\Gamma_0(n) = \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \pmod{n} \text{ is upper triangular}\}.$$

We have seen that $\Gamma_0(n) < \mathrm{SL}_2(\mathbb{Z})$. Prove that $\Gamma(n) \lhd \mathrm{SL}_2(\mathbb{Z})$ but $\Gamma_0(n) \ntriangleleft \mathrm{SL}_2(\mathbb{Z})$.