Modern Algebra
Spring 2025

Assignment 7.1
Due March 19

**Exercise 1.** Let $G, H$ be groups, $f : G \to H$ a homomorphism.

**a.** If $K \lhd H$, prove that $f^{-1}(K) \lhd G$.

**b.** If $K \lhd G$ and $f$ is surjective, prove that $f(K) \lhd H$.

**Exercise 2.** Let $G$ be a group, $H < G$ and $N \lhd G$.

**a.** Prove that $HN < G$.

**b.** Prove that $H \cap N \lhd H$.

**c.** Prove that if $H \lhd G$ and $H \cap N$ is trivial, then $hn = nh$ for all $h \in H$ and $n \in N$.

**Exercise 3.** Lang, II.4.28

**Exercise 4.** Prove that $\mathbb{Q}$ has no proper subgroups of finite index.

**Exercise 5.** Let $G$ be a group and $H < G$. The *normalizer of $H$ in $G$* is

$$N_G(H) = \{x \in G \mid xHx^{-1} = H\}.$$

**a.** Prove that $N_G(H)$ is a subgroup of $G$ containing $H$, and that $H$ is normal in $N_G(H)$.

**b.** Prove that the set $\{xHx^{-1} \mid x \in G\}$ of conjugates of $H$ is in one to one correspondence with the left cosets of $N_G(H)$ in $G$.