Modern Algebra
Spring 2025

Assignment 7.2
Due March 19

Exercise 1. Let $d, n \in \mathbb{N}$ with $d|n$ (so that $n\mathbb{Z} \subseteq d\mathbb{Z}$). Use the First Isomorphism Theorem to prove that

$$d\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(\tfrac{n}{d})\mathbb{Z}.$$

Exercise 2. Use the map $z \mapsto \frac{z}{|z|}$ and the First Isomorphism Theorem to show that

$$\mathbb{C}^\times/\mathbb{R}^+ \cong S^1.$$

In an analogous manner, show that

$$\mathbb{C}^\times/S^1 \cong \mathbb{R}^+$$

Exercise 3. Let G be a group.

a. Let $K \lhd G$. Prove that G/K is abelian if and only if $[G, G] < K$. [*Suggestion.* To show $[G, G] < K$ it suffices to prove that $[a, b] \in K$ for all $a, b \in G$. Why?]

b. Let $f : G \to H$ be a group homomorphism. Use part **a** and the First Isomorphism Theorem to prove that if H is abelian, then $[G, G] < \ker f$. This shows that $G/[G, G]$ is the largest abelian quotient (homomorphic image) of G.

Exercise 4. Let G be a group, $H < G$ and $N \lhd G$.

a. Prove that $NH = HN < G$.

b. Use the First Isomorphism Theorem to prove that $HN/N \cong H/(H \cap N)$.

Exercise 5. Lang, II.4.29(ab) (part (c) follows from the next exercise).

Exercise 6. Lang, II.4.30