Modern Algebra
Spring 2025

Assignment 8.1
Due March 26

Exercise 1. Let $\{H_i \mid i \in I\}$ be an indexed set of groups. If G is a group and for each $i \in I$ we have a homomorphism $f_i : G \to H_i$, show that the function

$$F : G \to \prod_{i \in I} H_i$$

given by $F(g) = (f_i(g))_{i \in I}$ is a homomorphism. What is its kernel?

Exercise 2. Let $m, n \in \mathbb{N}$ and suppose $\gcd(m, n) = 1$. In class we proved that there is a well-defined isomorphism

$$\mathbb{Z}/mn\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})$$

given by $a + mn\mathbb{Z} \mapsto (a + m\mathbb{Z}, a + n\mathbb{Z})$, a result I called the *Chinese Remainder Theorem.* Show this implies that given any $r, s \in \mathbb{Z}$ there exists a solution $x \in \mathbb{Z}$ to the system of simultaneous congruences

$$x \equiv r \pmod{m},$$
$$x \equiv s \pmod{n},$$

which is unique, up to the addition of multiples of mn.

Exercise 3. Prove that a finite group G with prime order must be cyclic, and can be generated by any of its nonidentity elements. [*Suggestion.* Choose $e \neq g \in G$ (why can this be done?), and apply Lagrange's Theorem to the subgroup of G generated by g.]