Modern Algebra
Spring 2025

Assignment 9.2
Due April 1

Exercise 1. Compute the sign of each permutation in Exercise 8.3.1.

Exercise 2. For $i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, n\}$, use induction on $r \geq 2$ to prove that

$$(i_1\, i_2\, \cdots i_r) = (i_1\, i_r)(i_1\, i_{r-1}) \cdots (i_1\, i_3)(i_1\, i_2).$$

Exercise 3. Since every permutation in S_n can be written as a product of transpositions, it follows that S_n is generated by the set of all transpositions:

$$S_n = \langle (i\, j) \mid 1 \leq i < j \leq n \rangle.$$

There are $\binom{n}{2} = \frac{n(n-1)}{2}$ transpositions total. It turns out this is roughly $n/2$ times more than we need.

- **a.** Show that S_n is generated by the $n-1$ transpositions $(1\,2), (1\,3), (1\,4), \ldots, (1\,n)$. [*Suggestion.* If $i \neq j$, conjugate $(1\, j)$ by $(1\, i)$.]
- **b.** Show that S_n is generated by the $n-1$ transpositions $(1\,2), (2\,3), (3\,4), \ldots, (n-1\ n)$. [*Suggestion.* Starting with $j = 2$, conjugate $(j-1\ j)$ by $(1\ j-1)$.]

Exercise 4. Show that $S_n = \langle (1\,2), (1\,2\,3\, \cdots\, n) \rangle$. [*Suggestion.* Let $\sigma = (1\,2\,3\, \cdots\, n)$ and $\tau = (1\,2)$. Let $\tau_1 = \tau$ and recursively define $\tau_{k+1} = \sigma \tau_k \sigma^{-1}$ for $k \geq 1$. Apply exercises 9.1.3 and part **b** above.]