Name: Key

Math 1311
Final Exam
Fall 2004

Please write clear, complete solutions with all work shown where required. No credit will be given to unsubstantiated results. Be neat and orderly. Each solution is worth 10 points.

1. In each case, find the limit if it exits. If the limit does not exist, explain why.

   (a) $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right) &= \lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\\ &= \lim_{x\to 0}\frac{\sin x - x\cos x}{x\sin x} \qquad \frac{0}{0}\\ &\overset{\mathscr{L}}{=} \lim_{x\to 0}\frac{\cos x + x\sin x - \cos x}{x\cos x+\sin x} \qquad \frac{0}{0}\\ &\overset{\mathscr{L}}{=} \lim_{x\to 0}\frac{x\cos x+\sin x}{-x\sin x+\cos x+\cos x}\\ &= \frac{0}{2}=0\end{aligned}$$

   (b) $\lim_{x\to 0}\frac{\sin 3\pi x}{\sin 2\pi x}$

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3\pi x}{\sin 2\pi x} &= \lim_{x\to 0}\frac{\sin 3\pi x}{3\pi x}\cdot\frac{2\pi x}{\sin 2\pi x}\cdot\frac{3\pi x}{2\pi x}\\ &= \frac{3}{2}\end{aligned}$$

2. In each case, find the value of the limit if it exists. If the limit does not exist, explain why.

   (a) $\lim_{h\to 0}\frac{f(0.5+h)-f(0.5)}{h}$ where $f(x)=\cos\pi x$

   $\lim_{h\to 0}\frac{f(0.5+h)-f(0.5)}{h} = f'(0.5) = -\pi\sin\frac{\pi}{2} = -\pi$

(b) $\lim_{x\to 1^+} (x-1)^{\ln x}$

$$\begin{aligned}
y &= (x-1)^{\ln x} \\
\ln y &= \ln x \ln(x-1) \\
\lim_{x\to 1^+} \ln y &= \ln\left(\lim_{x\to 1^+} y\right) \\
&= \lim_{x\to 1^+} \frac{\ln(x-1)}{\frac{1}{\ln x}} \\
\mathscr{L} & \\
&= \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} \\
\mathscr{L} & \\
&= \lim_{x\to 1^+} \frac{x(\ln x)^2}{x-1} \\
&= (\ln x)^2 + \frac{2x\ln x}{x} \\
&= 0 \\
\lim_{x\to 1^+} y &= e^0 = 1
\end{aligned}$$

3. For each of the following:

(a) $f'(x)$ if $f(x) = \dfrac{5x^2+3x+1}{x^3}$

$$\begin{aligned}
f(x) &= \frac{5x^2+3x+1}{x^3} \\
&= \frac{5}{x} + \frac{3}{x^2} + \frac{1}{x^3} \\
f'(x) &= -5x^{-2} - 6x^{-3} - 3x^{-4} \\
&= \frac{-5}{x^2} - \frac{6}{x^3} - \frac{3}{x^4}
\end{aligned}$$

(b) $f'(x)$ if $f(x) = \dfrac{2x^3+1}{\cos x}$

$$\begin{aligned}
f'(x) &= \frac{\cos x \cdot 6x^2 + (2x^3+1)\sin x}{(\cos x)^2} \\
&= \frac{6x^2\cos x + (2x^3+1)\sin x}{\cos^2 x} \\
&= 6x^2\sec x + (2x^3+1)\tan x \sec x
\end{aligned}$$

4. For each of the following:

(a) $\frac{dy}{dx}$ when $x = 0$, where y is a function defined implicitly by $x^2 - y = 3xy^2 - 1$

$$2x - \frac{dy}{dx} = 6xy\,\frac{dy}{dx} + 3y^2$$
$$\frac{dy}{dx}(6xy+1) = 2x - 3y^2 \Rightarrow \frac{dy}{dx} = \frac{2x-3y^2}{6xy+1}$$
$$x = 0 \Rightarrow -y = -1 \Rightarrow y = 1$$
$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{-3}{1} = -3$$

(b) $h'(1)$ where $h(x) = f(g(x))$, $g(1) = 3$, $f'(3) = 4$, $f'(1) = 6$, and $g'(0) = 3$

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ h'(t) &= f'(g(1))g'(1) \\ &= f'(3)g'(1) \\ &= 4 \cdot 6 \\ &= 24 \end{aligned}$$

5. For each of the following:

(a) All antiderivatives of $(5x-7)^{10}$

$$\begin{aligned} \int (5x-7)^{10}\,dx &= \frac{(5x-7)^{11}}{11 \cdot 5} + c \\ &= \frac{(5x-7)^{11}}{55} + c \end{aligned}$$

(b) $\displaystyle\int \frac{2x-1}{(2x^2-2x+8)^4}\,dx$

$$\begin{aligned} \text{Let } u &= 2x^2 - 2x + 8 \\ du &= (4x-2)\,dx \\ \frac{1}{2}\,du &= (4x-1)\,dx \\ \int \frac{2x-1}{(2x^2-2x+8)^4}\,dx &= \frac{1}{2}\int u^{-4}\,du \\ &= \frac{1}{2}\,\frac{u^{-3}}{-3} + c \\ &= -\frac{1}{6(2x^2-2x+8)^3} + c \end{aligned}$$

6. Find each of the following:

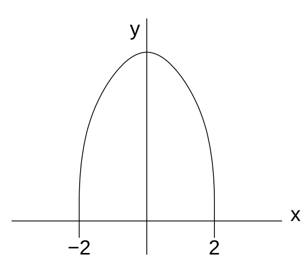
(a) $\int 3x\sqrt{x^2+1}\,dx$

$$
\begin{aligned}
\text{Let } u &= x^2+1\\
du &= 2x\ dx\\
\frac{1}{2}\ du &= x\ dx\\
\frac{3}{2}\int u^{\frac{1}{2}}\ du &= \frac{3}{2}\ \frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c\\
&= (x^2+1)^{\frac{3}{2}}+c
\end{aligned}
$$

(b) $\int_0^{\pi} \cos^3 x \sin x\ dx$

$$
\begin{aligned}
u &= \cos x,\\
du &= -\sin x\ dx\\
-du &= \sin x\ dx\\
&= -\int_0^{\pi} u^3\ du\\
&= \left.\frac{-u^4}{4}\right]_0^{\pi}\\
&= \left.-\frac{\cos^4 x}{4}\right]_0^{\pi}\\
&= -\frac{\cos^4 \pi}{4}+\frac{\cos^4 0}{4}\\
&= -\frac{1}{4}+\frac{1}{4}\\
&= 0
\end{aligned}
$$

7. Sketch the region under the curve $y = 4 - x^2$ and the x-axis.



(a) Find $\sum_{i=1}^{n} f(x_i)\Delta x$

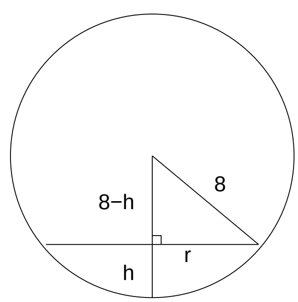
$$\Delta x = \frac{4}{n}, \quad x_i = -2 + \frac{4i}{n}, \quad f(x_i) = 4 - \left(-2 + \frac{4i}{n}\right)^2$$

$$\begin{aligned} f(x_i) &= 4 - \left[4 - \frac{16}{n}\,i + \frac{16}{n^2}\,i^2\right] \\ &= \frac{16}{n}\,i - \frac{16}{n^2}\,i^2 \\ \sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n}\left[\frac{16}{n}\,i - \frac{16}{n^2}\,i^2\right]\cdot\frac{4}{n} \\ &= \frac{64}{n^2}\sum_{i=1}^{n} i - \frac{64}{n^3}\sum_{i=1}^{n} i^2 \end{aligned}$$

(b) Find the area under the curve using the limit of the sum in part (a)

$$\begin{aligned} A &= \lim_{n\to\infty}\frac{64}{n^2}\,\frac{n(n+1)}{2} - \lim_{n\to\infty}\frac{64}{n^3}\,\frac{n(n+1)(2n+1)}{6} \\ &= 32 - \frac{64\cdot 2}{6} \\ &= \frac{32}{2} \end{aligned}$$

8. Suppose that the water is being emptied from a spherical tank of radius 8 ft. If the radius of the water in the tank is 6 ft and is decreasing at the rate of 2 ft/s, at what rate is the depth h of the water in the tank decreasing?



$$\begin{aligned} r &= 6 \\ (8-h)^2 &= 64 - 36 = 28 \\ (8-h) &= \sqrt{28} = 2\sqrt{7} \end{aligned}$$

$$r^2 + (8-h)^2 = 64$$

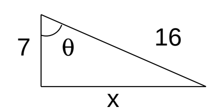
$$2r\,\frac{dr}{dt} + 2(8-h)\left(-\frac{dh}{dt}\right) = 0$$

$$\frac{dh}{dt} = \frac{2r}{2(8-h)}\,\frac{dr}{dt}$$

$$= -\frac{6}{2\sqrt{7}}(-2)$$

$$= \frac{6}{\sqrt{7}} \text{ ft/s}$$

9. There is a 16 ft ladder leaning against a vertical wall. The tip of the ladder is sliding down the wall at the rate of 5.6 ft/s. What is the rate of change, in radians per second, of the angle θ at the instant when the tip of the ladder is 7.0 ft above the ground?



$$x = \sqrt{16^2 - 7^2} = 3\sqrt{23}$$

$$\frac{dy}{dt} = -5.6 \text{ ft/s}$$

$$\cos\theta = \frac{y}{16}$$

$$-\sin\theta\,\frac{d\theta}{dt} = \frac{1}{16}\,\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{16\sin\theta}\,(-5.6)$$

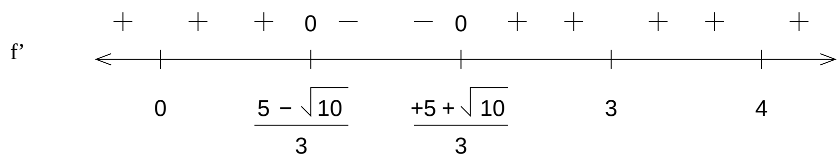
$$= -\frac{1}{16\left(\frac{3\sqrt{23}}{16}\right)}\,(-5.6)$$

$$= \frac{5.6}{3\sqrt{23}} \text{ radian/s}$$

10. Let f be a function given by $f(x) = x^3 - 5x^2 + 5x + k$, where k is a constant.

(a) On what intervals is f increasing? Find the critical points.

$$\begin{aligned} f'(x) &= 3x^2 - 10x + 5 \\ 0 &= 3x^2 - 10x + 5 \\ x &= \frac{10 \pm \sqrt{100 - 60}}{6} \\ &= \frac{+10 \pm 2\sqrt{10}}{6} \\ &= \frac{+5}{3} \pm \frac{1}{3}\sqrt{10} \end{aligned}$$



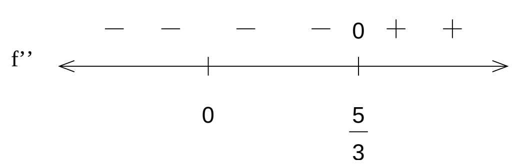
f is increasing at $\left(-\infty, \dfrac{5-\sqrt{10}}{3}\right) \cup \left(\dfrac{5+\sqrt{10}}{3}, \infty\right)$

Critical points $x = \dfrac{5-\sqrt{10}}{3}, \dfrac{5+\sqrt{10}}{3}$

(b) On what intervals is the graph of f concave downward? Find the inflection points.

$$f''(x) = 6x - 10$$

$$0 = 6x - 10 \Rightarrow x = \frac{10}{6} = \frac{5}{3}$$



f is concave downward on $\left(-\infty, \dfrac{5}{3}\right)$, and concave upward on $\left(\dfrac{5}{3}, \infty\right)$.

Inflection point $\left(\dfrac{5}{3}, \left(\dfrac{5}{3}\right)^3 - 5\left(\dfrac{5}{3}\right)^2 + 5\left(\dfrac{5}{3}\right) + k\right)$

(c) Find the value of k for which f has 11 as its relative minimum.

f has its relative minimum at $x = \dfrac{5+\sqrt{10}}{3}$

$$\begin{aligned}
f(x) = 11 &= \left(\frac{5+\sqrt{10}}{3}\right)^3 - 5\left(\frac{5+\sqrt{10}}{3}\right)^2 + 5\left(\frac{5+\sqrt{10}}{3}\right) + k \\
k &= 11 - \left(\frac{5+\sqrt{10}}{3}\right)\left[\frac{5}{3} - \frac{5}{3}(5+\sqrt{10}) + \frac{(5+\sqrt{10})^2}{9}\right] \\
&= 11 - \frac{(5+\sqrt{10})}{3}\left(\frac{55}{9} - \frac{5\sqrt{10}}{9}\right)
\end{aligned}$$