Math 1311
Pre Final Exam
Fall 2004

1. For each of the following, find the limit, it if exits. If the limit does not exist, explain why it does not.

   (a) $\lim_{x\to\sqrt{3}} \frac{x^2-3}{x-\sqrt{3}}$

$$\begin{aligned}\lim_{x\to\sqrt{3}} \frac{x^2-3}{x-\sqrt{3}} &= \lim_{x\to\sqrt{3}} \frac{(x-\sqrt{3})(x+\sqrt{3})}{x-\sqrt{3}}\\ &= 2\sqrt{3}\end{aligned}$$

   (b) $\lim_{x\to 0} \frac{\sin(5x)}{2x}$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(5x)}{2x} &= \lim_{x\to 0} \frac{5}{2}\cdot\frac{\sin 5x}{5x}\\ &= \frac{5}{2}\lim x \to 0 \frac{\sin 5x}{5x}\\ &= \frac{5}{2}\end{aligned}$$

   (c) $\lim\limits_{x\to\infty} (\ln x)^{\frac{1}{x}}$

$$\begin{aligned}y &= \lim_{x\to\infty} (\ln x)^{\frac{1}{x}}\\ \ln y &= \lim_{x\to\infty} \ln[\ln x]^{\frac{1}{x}}\\ &= \lim_{x\to\infty} \frac{1}{x}\ln(\ln x)\\ &\mathscr{L}\\ \ln y &= \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}\\ &= 0\\ y &= e^0 = 1\end{aligned}$$

2. For each of the following, find the limit, if it exists. If the limit does not exist, explain why it does not exist.

   (a) $\lim_{x\to 2} \frac{|x-2|}{x-2}$

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1$$
$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$$

   The limit does not exist since the limit from the left is not equal to the limit from the right.

   (b) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) &= \lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} \\
&\overset{\mathscr{L}}{=} \lim_{x\to 0} \frac{e^x - 1}{xe^x - e^x + 1} \\
&\overset{\mathscr{L}}{=} \lim_{x\to 0} \frac{e^x}{xe^x + e^x - e^x} \\
&= \lim_{x\to 0} \frac{1}{x} \\
&= \infty \\
y &= e^{\infty} = \infty
\end{aligned}$$

3. For each of the following, find the indicated derivative.

   (a) $f(x) = 3\sqrt{x} - \frac{1}{x} + 4$

$$\begin{aligned}
f(x) &= 3\sqrt{x} - \frac{1}{x} + 4 \\
&= 3x^{\frac{1}{2}} - x^{-1} + 4 \\
f'(x) &= \frac{3}{2}x^{-\frac{1}{2}} + \frac{1}{x^2} + 0 \\
&= \frac{3}{2\sqrt{x}} + \frac{1}{x^2}
\end{aligned}$$

(b) $y = \dfrac{\cos x}{1-x}$, find $\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+x)(-\sin x) - (-1)\cos x}{(1-x)^2} \\ &= \frac{\cos x - \sin x(1-x)}{(1-x)^2}\end{aligned}$$

(c) $y = 3\cos(2x)$, find $\dfrac{d^2y}{dx^2}$

$$\begin{aligned}y' &= -6\sin(2x) \\ y'' &= -12\cos(2x)\end{aligned}$$

4. For each of the following, find the indicated derivative.

(a) $xy^2 - y = x^2$, find $\dfrac{dy}{dx}$ when $x = 1$

$$\begin{aligned}x \cdot 2y\frac{dy}{dx} + y^2 - \frac{dy}{dx} &= 2x \\ \frac{dy}{dx}[2xy-1] &= 2x - y^2 \\ \frac{dy}{dx} &= \frac{2x-y^2}{2xy-1} \\ \frac{dy}{dx} &= \frac{2\cdot 1 - \left(\frac{1\pm\sqrt{5}}{2}\right)^2}{2\cdot 1\left(\frac{1\pm\sqrt{5}}{2}\right) - 1} \\ x = 1 &\Rightarrow \\ y^2 - y &= 1 \\ y^2 - y - 1 &= 0 \\ y &= \frac{1\pm\sqrt{1+4}}{2} \\ x = 1, \quad y &= \frac{1\pm\sqrt{5}}{2}\end{aligned}$$

(b) $h(x) = f(g(3x))$, find $h(2)$ if $g(6) = 1$, $f(1) = 4$, $f'(1) = \frac{1}{2}$, and $g'(6) = 3$

$$\begin{aligned} h'(x) &= f'(g(3x)) \cdot g'(3x) \cdot 3 \\ h'(2) &= f'(g(6))g'(6) \cdot 3 \\ &= f'(1)g'(6) \cdot 3 \\ &= \frac{1}{2} \cdot 3 \cdot 3 \\ &= \frac{9}{2} \end{aligned}$$

5. (a) If $\int_0^k (2kx - x^2)\, dx = 18$, find k

$$\begin{aligned} 2k\frac{x^2}{x} - \frac{x^3}{3}\Big]_0^k &= 18 \\ \frac{2k^3}{2} - \frac{k^3}{3} &= 18 \\ \frac{2}{3}k^3 &= 18 \\ k^3 &= 27 \\ k &= 3 \end{aligned}$$

(b) Find all antiderivatives (or the indefinite integral) of $5\sin(3x)$

$$\int 5\sin(3x)\, dx = ?$$

$$\begin{aligned} \text{Let} \quad u &= 3x, \\ du &= 3\, dx \\ \frac{1}{3}du &= dx \\ \frac{5}{3}\int \sin u\, du &= \frac{5}{3}(-\cos u) + c \\ &= -\frac{5}{3}\cos 3x + c \end{aligned}$$

(c) Find all antiderivatives (or the indefinite integral) of $\dfrac{x}{\sqrt{3x^2+5}}$

$$\int \frac{x}{(3x^2+5)^{\frac{1}{2}}}\,dx = ?$$

$$\begin{aligned}
\text{Let} \quad u &= 3x^2+5, \\
du &= 6x\,dx \\
\frac{1}{6}du &= x\,dx \\
\frac{1}{6}\int \frac{du}{u^{\frac{1}{2}}} &= \frac{1}{6}\int u^{-\frac{1}{2}}\,du \\
&= \frac{1}{3}(3x^2+5)^{\frac{1}{2}} + c
\end{aligned}$$

6. (a) Find all the antiderivatives (or indefinite integral) of $\dfrac{5x}{2x^2-1}$

$$u = 2x^2 - 1 \Rightarrow du = 4x\,dx \Rightarrow \frac{1}{4}\,du = x\,dx$$

$$\begin{aligned}
\int \frac{5x}{2x^2-1}\,dx &= \frac{5}{4}\int \frac{du}{u} \\
&= \frac{5}{4}\ln|u| + c \\
&= \frac{5}{4}\ln|2x^2-1| + c
\end{aligned}$$

(b) Find all antiderivatives (or the indefinite integral) of $(3x-5)^6$

$$u = 3x - 5 \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$$

$$\begin{aligned}
\int (3x-5)^6\,dx &= \frac{1}{3}\int u^6\,du \\
&= \frac{1}{3}\frac{u^7}{7} + c \\
&= \frac{1}{21}(3x-5)^7 + c
\end{aligned}$$

(c) If $\int_1^5 f(x)\,dx = 4$ and $\int_3^5 f(x)\,dx = -7$, then what is $\int_1^3 f(x)\,dx$?

$$\begin{aligned}
\int_1^3 f(x) = dx &= \int_1^5 f(x)\,dx - \int_3^5 f(x)\,dx \\
&= 4 - (-7) \\
&= 11
\end{aligned}$$

7. (a)

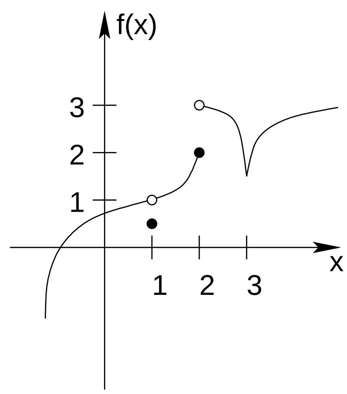
$$f(x) = \begin{cases} 3x+1 & \text{if } x \le 2 \\ cx^2 & \text{if } x > 2 \end{cases}$$

Is there a value of c for which f is continuous on $(-\infty, \infty)$? If so, find it. If not, explain why there is no such c.

$$\begin{aligned} f(2) = 3 \cdot 2 + 1 &= 7 \\ x = 2, \quad cx^2 &= 7 \\ 4c &= 7 \\ c &= \frac{7}{4} \end{aligned}$$

If $c = \frac{7}{4}$, f is continuous.

(b)



i. For which values of x is f discontinuous?

$$x = 1, \quad x = 2$$

ii. For which values of x is f not discontinuous?

$$x = 1, \quad x = 2, \quad x = 3$$

8. (a) Find $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$ if $f(x) = x^3 - 4$.

$$\begin{aligned} f'(x) &= 3x^2 \\ \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} &= f'(3) \\ &= 3(9) \\ &= 27 \end{aligned}$$

(b) Find $\lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$ if $f(x) = x^3 - 4$.

$$\begin{aligned}\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} &= f'(3) \\ &= 27\end{aligned}$$

9. Let $F(x) = \int_0^x \sqrt{t^3 - 1}\, dt$.

   (a) Find $F'(x)$.

$$F'(x) = \sqrt{x^3 - 1}$$

   (b) Find $\lim\limits_{x \to 0} F(x)$.

$$\lim_{x \to 0} F(x) = 0$$

10. A function f that is continuous for all real numbers x has $f(3) = -1$ and $f(7) = 1$. If $f(x) = 0$ for exactly one value of x, then which of the following could be x? Justify your answer.

    a) -1 b) 0 c)1 d) 4 e) 9

    d) 4

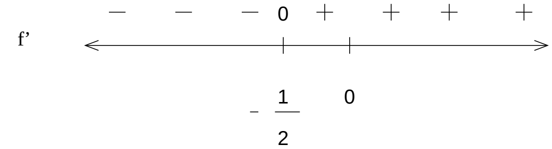
11. If $f'(x) = x^2 + x - 12$, then f is increasing on

    (a) $(-4, 3)$
    (b) $(-3, 4)$
    (c) $\left(-\infty, -\frac{1}{3}\right)$
    (d) $(-\infty, -4)$ and $(3, \infty)$
    (e) None of the above.

    Justify your answer.

(e) None of the above.

$$f'(x) = 2x + 1,$$



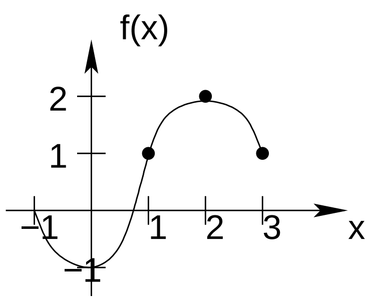
$$
\begin{aligned}
f'(x) > 0 \quad \text{if} \quad 2x + 1 &> 0 \\
2x &> -1 \\
x &> -\frac{1}{2}
\end{aligned}
$$

12. Suppose that $f(1) = 0$ and that $1 \le f'(x) \le 2$ for x in $[0, 4]$. Use the Mean Value Theorem to explain why $f(4)$ cannot be 10.

$$
\begin{aligned}
&\text{If} \quad f(4) = 10, \quad \text{then by the Mean Value Theorem} \\
&\frac{f(4) - f(1)}{4 - 1} = f'(c) \quad \text{where } c \text{ is between 1 and 4} \\
&\frac{10 - 0}{3} = f'(c) \\
&\frac{10}{3} = f'(c)
\end{aligned}
$$

But this is impossible since $1 \le f' \le 2$. Thus $f(4) \ne 10$.

13.



(a) What is the average rate of change of f on $[-1, 3]$?

$$
\begin{aligned}
\text{Average rate} &= \frac{f(3) - f(-1)}{3 - (-1)} \\
&= \frac{1 - 0}{4} = \frac{1}{4}
\end{aligned}
$$

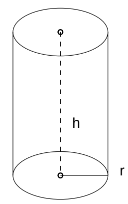
(b) On what intervals is $f'(x)$ increasing?

$$(-1, 1)$$

(c) On what intervals is $f'(x)$ decreasing?

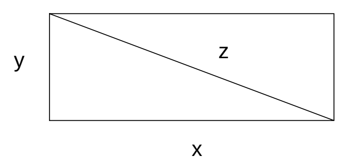
$$(1, 3)$$

14. The volume of a cylindrical tin can with a top and bottom is to be 18π cubic inches. If a minimal amount of tin is to be used to construct the can, what must be the height, in inches, of the can? (You may want to know that the surface area of a cylinder, excluding a top and bottom, is $2\pi rh$.).



$$\begin{aligned} v &= 18\pi \\ \pi r^2 h &= 18\pi \\ h &= \frac{18}{r^2} \\ A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r\left(\frac{18}{r^2}\right) \\ &= 2\pi r^2 + 36\pi r^{-1} \\ A' &= 4\pi r - \frac{36\pi}{r^2} = 0 \\ r^3 &= 9 \Rightarrow r = \sqrt[3]{9}, \quad h = \frac{18}{\sqrt[3]{81}} \end{aligned}$$

15. The sides of the rectangle increase in such a way that $\frac{dz}{dt} = \frac{1}{2}$ and $\frac{dx}{dt} = 2\frac{dy}{dt}$. At the instant when $x = 3$ and $y = 2$, what is the value of $\frac{dx}{dt}$?



$$
\begin{aligned}
z^2 &= x^2 + y^2 \\
2z\,\frac{dz}{dt} &= 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \\
\sqrt{13}\left(\frac{1}{2}\right) &= 3\,\frac{dx}{dt} + 2 \cdot \frac{1}{2}\,\frac{dx}{dt} \\
\frac{\sqrt{13}}{2} &= 4\,\frac{dx}{dt} \\
\frac{dx}{dt} &= \frac{\sqrt{13}}{8}
\end{aligned}
$$