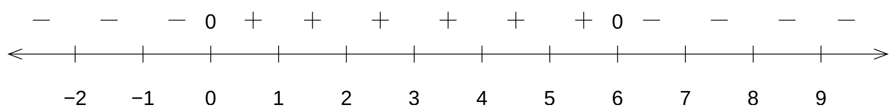
Math 1311
Test 2
Fall 2004
SOLUTIONS

1. (a) (10 points) Apply the first derivative test to classify each of the critcal points of the function $f(x) = x^2 e^{-x/3}$. If you have a graphics calculator, plot $y = f(x)$ to see whether the appearance of the graph corresponds to your classification of the critical points.

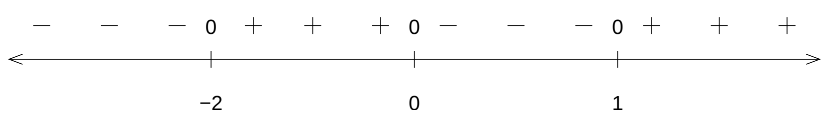
$$f'(x) = x^2 \left(-\frac{1}{3}\right) e^{-\frac{x}{3}} + 2xe^{-\frac{x}{3}}$$
$$= xe^{-\frac{x}{3}} \left[-\frac{1}{3}x + 2\right]$$
$$\text{critical points:} \quad f' = 0 \Rightarrow x_1 = 0, \quad x_2 = 6$$
$$-\frac{1}{3}x + 2 = 0$$



The critical point $x = 0$ gives a local minimum, while at $x = 6$, f has a local maximum.

(b) (10 points) Determine the open intervals on the x-axis on which the function $f(x) = 3x^4 + 4x^3 - 12x^2$ is increasing as well as those on which it is decreasing. If you have a graphics calculator, plot the graph $y = f(x)$ to see whether it agrees with your result.

$$f'(x) = 12x^3 + 12x^2 - 24x$$
$$= 12x(x^2 + x - 2)$$
$$= 12x(x + 2)(x - 1)$$
$$\text{critical points are} \quad -2, \quad 0, \quad 1$$



f is increasing on $(-2, 0) \cup (1, \infty)$
f is decreasing on $(-\infty, -2) \cup (0, 1)$

2. Find $\dfrac{dy}{dx}$

(a) (10 points) $y = e^{-2x} \sin 3x$

$$\begin{aligned} y' &= e^{-2x}(\cos 3x) \cdot 3 + (-2)e^{-2x} \sin 3x \\ &= 3 \cos 3x e^{-2x} - 2 \sin 3x e^{-2x} \\ &= e^{-2x}[3 \cos 3x - 2 \sin 3x] \end{aligned}$$

(b) (10 points) $x \ln y = x + y$

$$\begin{aligned} x \cdot \frac{1}{y} \frac{dy}{dx} + \ln y &= 1 + \frac{dy}{dx} \\ \frac{dy}{dx}\left[\frac{x}{y} - 1\right] &= 1 - \ln y \\ \frac{dy}{dx} &= \frac{1 - \ln y}{\frac{x}{y} - 1} \\ &= \frac{y - y \ln y}{x - y} \end{aligned}$$

3. (a) (10 points) Write an equation of the line tangent to the given curve at $x^2 - 3xy + 2y^2 = 0$.

$$\begin{aligned} 2x - 3x\frac{dy}{dx} - 3y + 4y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}[4y - 3x] &= 3y - 2x \\ \frac{dy}{dx} &= \frac{3y - 2x}{4y - 3x} \end{aligned}$$

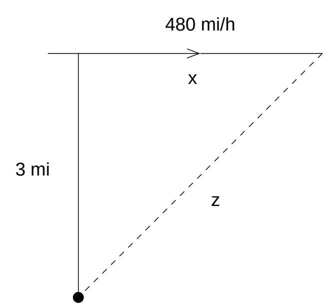
at (x_1, y_1), the equation of the tangent line is

$$\begin{aligned} y - y_1 &= \frac{3y_1 - 2x_1}{4y_1 - 3x_1}(x - x_1) \\ y &= y_1 + \frac{3y_1 - 2x_1}{4y_1 - 3x}(x - x_1) \end{aligned}$$

(b) (10 points) Find $\dfrac{dy}{dx}$ if $y = \left(1 + \dfrac{1}{x}\right)^x$

$$\begin{aligned}
\ln y &= \ln\left(1 + \frac{1}{x}\right)^x \\
\ln y &= x \ln\left(1 + \frac{1}{x}\right) \\
\frac{1}{y}\frac{dy}{dx} &= x \cdot \frac{1}{1 + \frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) + \ln\left(1 + \frac{1}{x}\right) \\
&= \frac{-x}{x^2 + x} + \ln\left(1 + \frac{1}{x}\right) \\
\frac{dy}{dx} &= y\left[\frac{-1}{x+1} + \ln\left(1 + \frac{1}{x}\right)\right] \\
&= \left(1 + \frac{1}{x}\right)^x \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}\right]
\end{aligned}$$

4. (20 points) An airplane flying horizontally at an altitude of 3 mi and at a speed of 480 mi/h passes directly above an observer on the ground. How fast is the distance from the observer to the airplane increasing 30 s later?



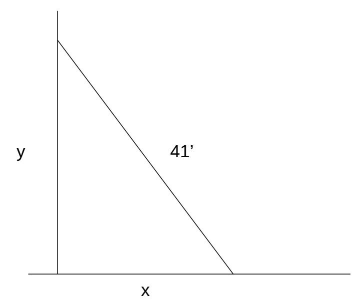
First Method

$$\begin{aligned}
z^2 &= x^2 + 9 \\
\text{But} \quad x &= 480t \\
z^2 &= (480)^2 t^2 + 9 \\
2z\,\frac{dz}{dt} &= 2(480)^2 t \\
\frac{dz}{dt} &= \frac{(480)^2 t}{z} \\
&= \frac{(480)^2}{5} \cdot \frac{1}{120} \\
&= 4.96 = 384 \text{ mi/h}
\end{aligned}$$

Second Method

$$
\begin{aligned}
z^2 &= x^2 + 9 \\
2z \frac{dz}{dt} &= 2x \frac{dx}{dt} \\
\frac{dz}{dt} &= \frac{x}{z} \frac{dx}{dt} \\
\text{Afer 30 seconds,} \quad x &= 480 \cdot \left(\frac{1}{120}\right) \\
z &= \sqrt{4^2 + 3^2} = 5 \\
\frac{dz}{dt} &= \frac{4}{5} \cdot 480 \\
&\approx 384 \text{ mi/h}
\end{aligned}
$$

5. (20 points) A ladder 41 ft long that was leaning against a vertical wall begins to slip. Its top slides down the wall while its bottom moves along the level ground at a constant speed of 4 ft/s. How fast is the top of the ladder moving when it is 9 ft above the ground?



$$
\begin{aligned}
\frac{dx}{dt} &= 4\text{ft/s} \\
y^2 &= (41)^2 - x^2 \\
2y \frac{dy}{dt} &= -2x \frac{dx}{dt} \\
\frac{dy}{dt} &= -\frac{2x}{2y} \frac{dx}{dt}
\end{aligned}
$$

$$\text{If } y = 9', \quad \text{then } x = \sqrt{(41)^2 - (9)^2} = 40'.$$

$$\text{Hence } \frac{dy}{dt} = -\frac{40}{9} \cdot 4 = -\frac{160}{9} \text{ft/s.}$$