Name: Key

Math 1311
Test 3
Fall 2004

1. Solve the initial value problem in $\dfrac{dy}{dx} = \dfrac{1}{\sqrt[3]{x}}$; $y(1) = 1$.

$$\begin{aligned} y &= \int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-\frac{1}{3}}\,dx \\ y &= \frac{3}{2}x^{\frac{2}{3}} + c \\ y(1) = 1 &= \frac{3}{2} + c \Rightarrow c = -\frac{1}{2} \\ y(x) &= \frac{3}{2}x^{\frac{2}{3}} - \frac{1}{2} \end{aligned}$$

2. First calculate (in terms of n) the sum

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

to approximate the area A of the region under $f(x) = x^3$ above the interval $[0, 3]$. Then find A exactly by taking the limit as $n \to \infty$.

$$\Delta x = \frac{3}{n} \quad x_i = \frac{3i}{n}, \quad f(x_i) = \frac{27}{n^3}i^3$$

$$\begin{aligned} \sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \frac{27}{n^3}i^3 \cdot \frac{3}{n} \\ &= \frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 \end{aligned}$$

$$A = \lim_{n\to\infty} \frac{81}{n^4}\frac{n^2(n+1)^2}{4} = \frac{81}{4}$$

3. Find the limit of $\lim_{x\to\pi} \frac{1+\cos x}{(x-\pi)^2}$. $\left(\frac{0}{0}\right)$
By L'Hopital Rule

$$\lim_{x\to\pi} \frac{1+\cos x}{(x-\pi)^2} = \lim_{x\to\pi} \frac{-\sin x}{2(x-\pi)} \quad \left(\frac{0}{0}\right)$$

Applying L'Hopital Rule again yields

$$\begin{aligned} &= \lim_{x\to\pi} \frac{-\cos x}{2} \\ &= \frac{1}{2} \end{aligned}$$

4. Find the limit of $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{1-\cos x}\right)$. $(\infty - \infty)$

$$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{1-\cos x}\right) = \lim_{x\to 0} \frac{1-\cos x - x^2}{x^2(1-\cos x)} \quad \left(\frac{0}{0}\right)$$

Applying L'Hopital Rule yields

$$= \lim_{x\to 0} \frac{\sin x - 2x}{x^2 \sin x + 2x(1-\cos x)} \quad \left(\frac{0}{0}\right)$$

$\mathscr{L}$

$$\begin{aligned} &= \lim_{x\to 0} \frac{\cos x - 2}{x^2 \cos x + 2x\sin x + 2x \sin x + 2(1-\cos x)} \\ &= \frac{-1}{0} \\ &= -\infty \end{aligned}$$

5. Sketch the graph of $f(x) = \dfrac{x^3}{x^2-1}$ indicating all critial points, inflection points, and asymptotes. Show the concave structure clearly.

$$\text{Asymptotes: } \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1} \rightarrow y = x$$

$$\text{Vertical asymptotes: } x = 1, \quad x = -1$$

$$f'(x) = \frac{3x^2(x^2-1) - 2x \cdot x^3}{(x^2-1)^2}$$

$$= \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2}$$

$$= \frac{x^2(x^2-3)}{(x^2-1)^2}$$

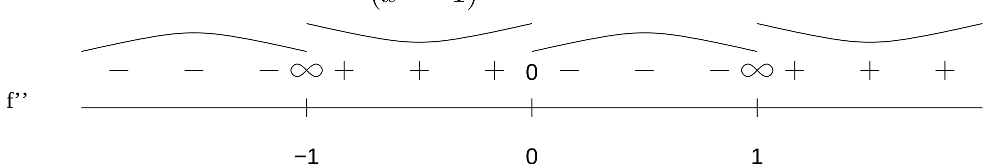
$$\text{Critical points: } x = 0, \quad x = \sqrt{3}, \quad x = -\sqrt{3}, \quad x = 1, \quad x = -1$$

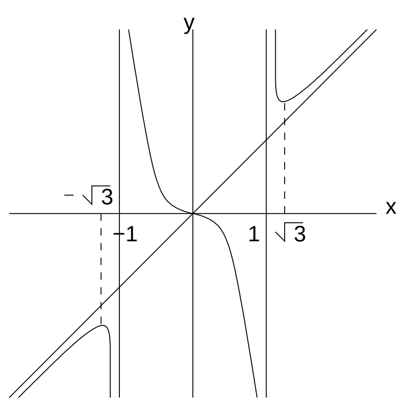
$$f''(x) = \frac{(x^2-1)^2(4x^3-6x) - 2 \cdot 2x(x^2-1)(x^4-3x^2)}{(x^2-1)^4}$$

$$= \frac{4x^5 - 6x^3 - 4x^3 + 6x - 4x^5 + 12x^3}{(x^2-1)^3}$$

$$= \frac{2x^3+6x}{(x^2-1)^3}$$

$$= \frac{2x(x^2+3)}{(x^2-1)^3}$$





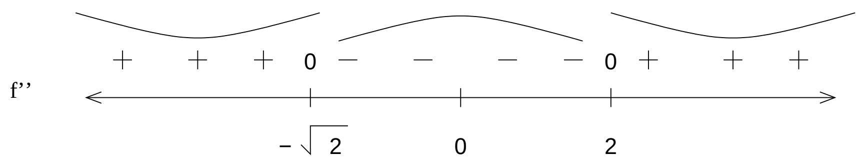
6. Sketch the graph of $f(x) = x^4 - 12x^2$ indicating all critial points, inflection points, and asymptotes. Show the concave structure clearly.

$$f'(x) = 4x^3 - 24x = 4x(x^2 - 6)$$

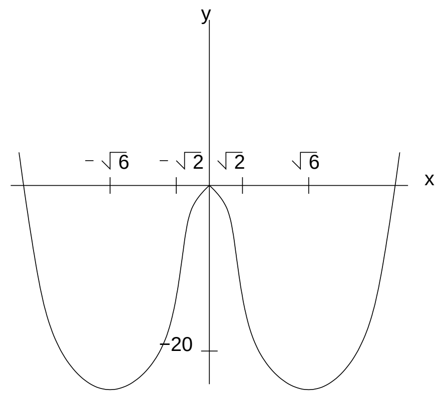
Critical points: $x = 0, \quad x = \sqrt{6}, \quad x = -\sqrt{6}$

There are no asymptotes

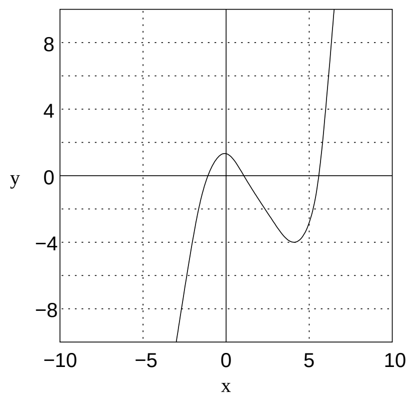
$$f''(x) = 12x^2 - 24 = 12(x^2 - 2)$$



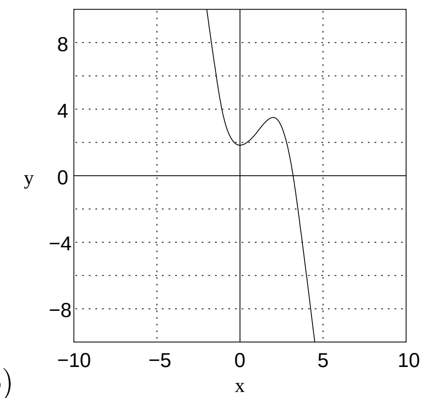
Inflection points: $(-\sqrt{2}, -20), \quad (\sqrt{2}, -20)$



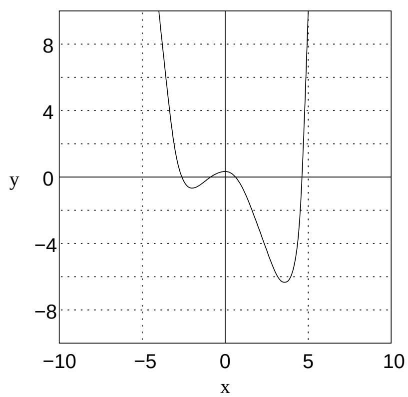
7. Match the graph of each function in (a)-(f) with the graph of its second derivative in I-VI.



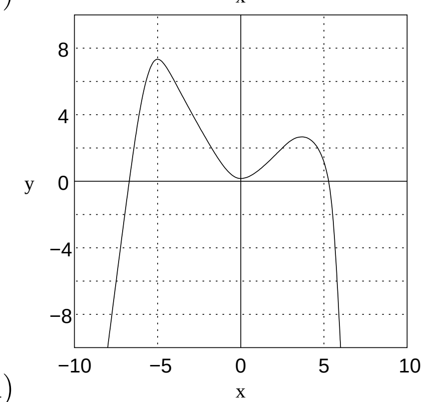
(a)



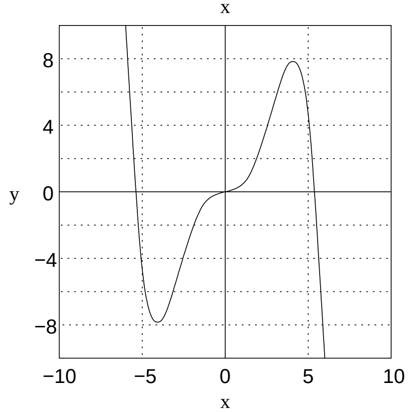
(b)



(c)

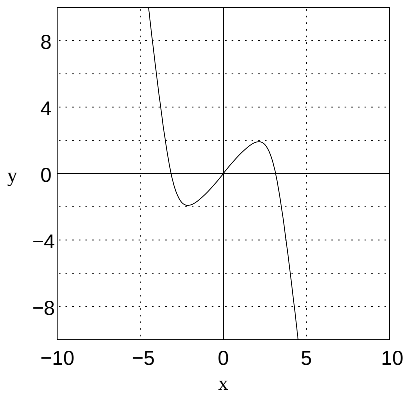


(d)



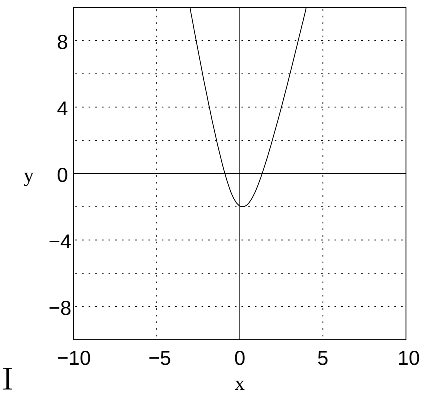
(e)

(a) $\rightarrow$ III
(b) $\rightarrow$ IV
(c) $\rightarrow$ II
(d) $\rightarrow$ V
(e) $\rightarrow$ I



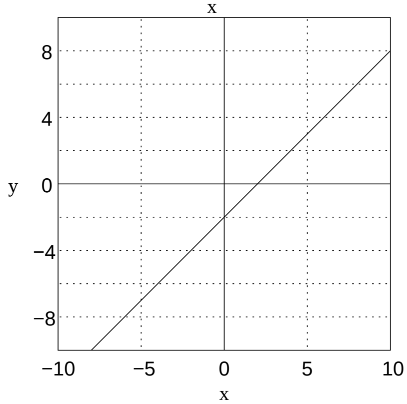
8
4
y 0
−4
−8
−10 −5 0 5 10
x

I



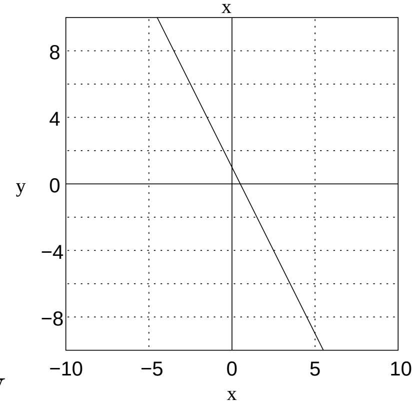
8
4
y 0
−4
−8
−10 −5 0 5 10
x

II



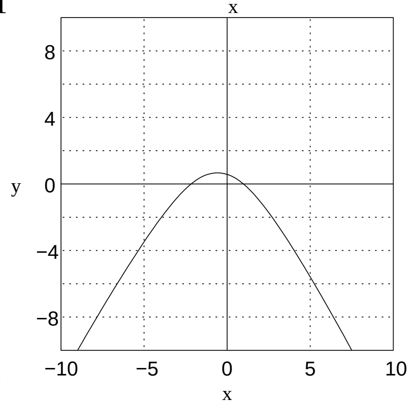
8
4
y 0
−4
−8
−10 −5 0 5 10
x

III



8
4
y 0
−4
−8
−10 −5 0 5 10
x

IV



8
4
y 0
−4
−8
−10 −5 0 5 10
x

V