Math 3336
Sprint 2005
Test I
Answers

1. Determine the values of a so that the following system in the unknowns x, y, z has:

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + az &= 3 \\ x + ay + 3z &= 2 \end{aligned}$$

(a) no solution,
Answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & a & 3 \\ 1 & a & 3 & 2 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & a-1 & 4 & 1 \end{array}\right] \xrightarrow{(1-a)R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0-a^2 & -a+6 & 2-a \end{array}\right] \xrightarrow{\frac{1}{-a^2-a+6}R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & 1 & \frac{2-a}{-a^2-a+6} \end{array}\right]$$

if $a \neq 2$. There is no solution if $-(a^2 + a - 6) = -(a + 3)(a - 2) = 0$ or $a = -3$ or $a = 2$ and $2 - a = 0$ $(a = 2)$. Thus $a = -3$ gives no solution.

(b) more than one solution,
Answer: If $a = 2$, then the last row in the echelon now consists of zeros. Hence one free variable which leads to many solutions.

(c) a unique solution: Answer: If $a \neq -3$ or 2, then we have a unique solution.

2. Consider the following matrix

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}$$

(a) Reduce A to a row-echelon form

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix} \xrightarrow[-4R_1+R_4]{-2R_1+R_3} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{pmatrix} \xrightarrow[R_2+R_4]{R_2+R_3} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Reduce A to a reduced row-echelon form

$$\xrightarrow{\frac{1}{11}R_2} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-3R_2 + R_1} \begin{pmatrix} 1 & 0 & \frac{4}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) If A^{-1} exists, find it.

$$A^{-1} \quad \text{does not exist.}$$

3. (a) Determine whether or not the set W of all vectors in $\mathbb{R}^4$ is a subspace of $\mathbb{R}^4$, where

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_2 = x_3 + x_4 \right\}$$

(i) Let $\vec{u} = (x_1, x_2, x_3, x_4)^T \in W$, $\vec{v} = (y_1, y_2, y_3, y_4)^T \in W$. Then $\vec{w} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)^T \in W$. Then $(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = (x_3 + x_4) + (y_3 + y_4) = (x_3 + y_3) + (x_4 + y_4)$ Therefore $\vec{u} + \vec{v} \in W$.

(ii) Let $\vec{u} \in W$, $c \in \mathbb{R}$. Thus $c\vec{u} = (cu_1, cu_2, cu_3)^T$. $cu_1 + cu_2 = c(u_1 + u_2) = c(u_3 + u_4) = cu_3 + cu_4$. Therefore W is a subspace of $\mathbb{R}^4$.

(b) Suppose A is an $n \times n$ matrix and that k is a constant scalar. Show that the set of all vectors $\vec{x}$ such that $A\vec{x} = k\vec{x}$ is a subspace of $\mathbb{R}^4$.
Let $W = \{\vec{x} : A\vec{x} = k\vec{x}\}$.

(i) Let $\vec{u}, \vec{v} \in W$. Then $A\vec{u} = k\vec{u}$ and $A\vec{v} = k\vec{v}$. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = k\vec{u} + k\vec{v} = k(\vec{u} + \vec{v})$.

(ii) $Ac\vec{u} = cA\vec{u} = ck\vec{u}$. $k(c\vec{u})$. Hence W is a subspace.

4. Solve the differential equation

$$xy' = 3y + x^4 \cos x, \quad y\left(\frac{\pi}{2}\right) = 0$$

$y' - \dfrac{3}{x}y = x^3 \cos x.$

Integrating factor $\mu(x) = e^{-\int \frac{3}{x}\,dx} = e^{-3\ln x} = e^{\ln \frac{1}{x^3}} = \dfrac{1}{x^3}$

$$\int d\left(\frac{1}{x^3}y\right) = \int \cos x \, dx$$

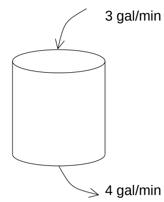
$$\frac{1}{x^3}y(x) = \sin x + c \Rightarrow y(x) = x^3 \sin x + cx^3$$

$$y\left(\frac{\pi}{2}\right) = 0 = \frac{\pi^3}{8} + \frac{\pi^3}{8}c \Rightarrow c = -1$$

$$y(x) = x^3(\sin x - 1)$$

5. A tank initially contains 20 gallons of pure water. Brine containing 2 pounds of salt per gallon enters the tank at 3 gal/min and the perfectly mixed solution leaves the tank at 4 gal/min.

(a) Find the amount of salt in the tank after t time.



Let $x(t)$ be the amount of salt in the tank at time t.

$$\frac{dx}{dt} = 3.2 - 4\frac{x(t)}{20-t}$$
$$\frac{dx}{dt} + \frac{4x(t)}{20-t} = 6$$
$$\mu(x) = e^{\int \frac{4}{20-t}\,dt}$$
$$= e^{-4\ln(20-t)}$$
$$= \frac{1}{(20-t)^4}$$
$$\frac{1}{(20-t)^4}x(t) = \int \frac{6}{(20-t)^4}\,dt = -6\frac{(20-t)^{-3}}{-3} + c$$
$$x(t) = 2(20-t) + c(20-t)^4, \quad x(0) = 0 = 40 + c(20)^4 \Rightarrow c = \frac{-40}{(20)^4}$$
$$x(t) = 2(20-t) - \frac{1}{4000}(20-t)^4$$

(b) Find the amount of salt in the tank after 15 minutes.

$$x(15) = 2(20-15) - \frac{1}{4000}(20-15)^4$$
$$= 10 - \frac{625}{4000}$$
$$= 9\frac{27}{32}\text{ lb}$$

6. Consider an animal population $P(t)$ with constant death rate $\delta = 0.01$ (deaths per animal per month) and with birth rate β proportional to P. Suppose that $P(0) = 200$, and $P'(0) = 2$.

(a) When $P = 1000$?

$$\frac{dp}{dt} = P(kP - 0.01), \quad P'(0) = 2 = 200(k(200) - 0.01)$$
$$\frac{dp}{dt} = P\left(\frac{P}{10000} - \frac{1}{100}\right)$$
$$= \frac{1}{10000}P(P - 100)$$
$$\int \frac{dp}{P(P-100)} = \int \frac{1}{10000}\, dt \Rightarrow \int \frac{1}{100}\left(\frac{1}{P-100} - \frac{1}{P}\right)\, dp = \int \frac{1}{10000}\, dt$$
$$\ln(P - 100) - \ln P = \frac{1}{100}t + \ln c$$
$$\frac{P-100}{P} = ce^{\frac{1}{100}t} \Rightarrow \frac{200-100}{200} = c \Rightarrow \frac{1}{2}$$
$$\frac{P-100}{P} = \frac{\frac{1}{2}e^{\frac{1}{100}t}}{1 - \frac{1}{2}e^{\frac{1}{100}t}} \Rightarrow P(t) = 100 + \frac{100}{2 - e^{-\frac{1}{100}t} - 1}$$
$$P(t) = 1000 = 100 + \frac{100}{2e^{-\frac{1}{100}t} - 1} \Rightarrow t = 100\ln\left(\frac{18}{10}\right) \approx 58.7$$

(b) When does doomsday occur? (Doomsday occurs when the population tends to ∞ in a finite time)
Doomsday occurs when

$$2e^{-\frac{1}{100}t} - 1 = 0$$
$$e^{-\frac{1}{100}t} = \frac{1}{2}$$
$$-\frac{1}{100}t = \ln\frac{1}{2} \Rightarrow t = 100\ln 2 \approx 69.3$$

7. (a) Find the critical points (equilibrium points/constant solutions) of the differential equation

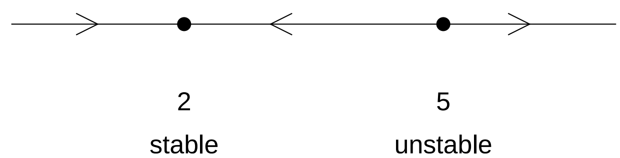
$$\frac{dx}{dt} = x^2 - 7x + 10.$$

Two critical points (equilibria) when $\frac{dx}{dt} = 0$

$$x^2 - 7x + 10 = 0$$
$$(x-5)(x-2) = 0$$
$$x(t) = 5, \quad x(t) = 2$$

(b) Determine the stability of the critical points without solving the equation.

$$\frac{dx}{dt} = (x-2)(x-5) \begin{cases} + & \text{if } x < 2 \Rightarrow x(t) \text{ is increasing,} \\ - & \text{if } 2 < x < 5 \Rightarrow x(t) \text{ is decreasing,} \\ + & \text{if } x > 5 \Rightarrow x(t) \text{ is increasing.} \end{cases}$$



(c) Draw a rough graph of the solution curves.