1. Suppose that
\[ I_n = [a_n, b_n], \quad I_{n+1} \subseteq I_n \text{ for each } n = 1, 2, 3, \ldots. \]
If \( \lim_{n \to \infty} (b_n - a_n) = 0 \), prove that there is only one number \( x_0 \) which is in every \( I_n \).

2. (a) Prove that if \( f \) is uniformly continuous on a bounded set \( S \), then \( f \) is a bounded function on \( S \).

(b) Use (a) to prove that \( f(x) = \frac{1}{x^2} \) is not uniformly continuous on \((0,1)\).

3. Which of the following functions are uniformly continuous on the specified set? Justify your answer by using any theorem in the notes or the book.
   (a) \( f(x) = x^2 \sin \frac{1}{x} \) on \((0,1]\)

   (b) \( f(x) = x^3 \) on \(\mathbb{R}\)
(c) \( f(x) = \sin\left(\frac{1}{x^2}\right) \) on \((0,1]\)

(d) \( f(x) = \frac{1}{x^3} \) on \((0,1]\)

4. (a) If \( p > 0 \) and \( \alpha \) is real, prove that \( \lim_{n\to\infty} \frac{n^\alpha}{(1 + p)^n} = 0 \)

(b) If \( |x| < 1 \), prove that \( \lim_{n\to\infty} x^n = 0 \)

5. Suppose that the functions \( f \) and \( g \) are uniformly continuous on a set \( S \). Prove that \( h = f + g \) is uniformly continuous on \( S \).
6. Let \( (s_n) \) be a bounded sequence in \( \mathbb{R} \). Let \( A \) be the set of \( a \in \mathbb{R} \) such that \( \{n \in \mathbb{N} : s_n < a\} \) is finite. Let \( B \) be the set of \( b \in \mathbb{R} \) such that \( \{n \in \mathbb{N} : s_n > b\} \) is finite. Prove that \( \sup A = \liminf s_n \) and \( \inf B = \limsup s_n \).

7. Let \( (s_n) \) be any sequence of nonzero real numbers. Prove that
\[
\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{\frac{1}{n}}.
\]

8. Determine which of the following series converge. You must justify your answer.
   (a) \[
   \sum \left[ \sin \left( \frac{n\pi}{6} \right) \right]^n
   \]
   (b) \[
   \sum \frac{(-1)^n n!}{2^n}
   \]
   (c) \[
   \sum \sqrt{n+1} - \sqrt{n}
   \]
9. Let \((a_n)\) be a sequence such that \(\liminf |a_n| = 0\). Prove that there is a subsequence \((a_{n_k})^\infty_{k=1}\) such that \(\sum_{k=1}^{\infty} a_{n_k}\) converges.

10. (a) Let \(a, b \in \mathbb{R}\). Prove that there are infinitely many rationales between \(a\) and \(b\).

(b) If \(r\) is rational \((r \neq 0)\), and \(x\) is irrational, prove that \(r + x\) and \(rx\) are irrational.