Name:________________________

Math 3360
Fall 2002
Take Home Final

1. Suppose that

$$I_n = [a_n, b_n],\ I_{n+1} \subset I_n \text{ for each } n = 1,2,3,\ldots.$$

If $\lim_{n\to\infty}(b_n - a_n) = 0$, prove that there is only one number x_0 which is in every I_n.

2. (a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

(b) Use (a) to prove that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0,1)$.

3. Which of the following functions are uniformly continuous on the specified set? Justify your answer by using any theorem in the notes or the book.

(a) $f(x) = x^2 \sin\frac{1}{x}$ on $(0,1]$

(b) $f(x) = x^3$ on $\mathbb{R}$

(c) $f(x) = \sin\left(\frac{1}{x^2}\right)$ on $(0,1]$

(d) $f(x) = \frac{1}{x^3}$ on $(0,1]$

4. (a) If $p > 0$ and α is real, prove that $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(b) If $|x| < 1$, prove that $\lim_{n\to\infty} x^n = 0$

5. Suppose that the functions f and g are uniformly continuous on a set S. Prove that $h = f + g$ is uniformly continuous on S.

6. Let (s_n) be a bounded sequence in $\mathbb{R}$. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathrm{N} : s_n < a\}$ is finite. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathrm{N} : s_n > b\}$ is finite. Prove that $\sup A = \liminf s_n$ and $\inf B = \limsup s_n$.

7. Let (s_n) be any sequence of nonzero real numbers. Prove that

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \le \liminf |s_n|^{\frac{1}{n}}.$$

8. Determine which of the following series converge. You must justify your answer.

(a) $\sum \left[\sin\left(\frac{n\pi}{6} \right) \right]^n$

(b) $\sum \frac{(-1)^n n!}{2^n}$

(c) $\sum \sqrt{n+1} - \sqrt{n}$

(d) $\sum \frac{1}{\sqrt{n!}}$

9. Let (a_n) be a sequence such that $\liminf |a_n| = 0$. Prove that there is a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

10. (a) Let $a, b \in \mathbb{R}$. Prove that there are infinitely many rationales between a and b.

(b) If r is rational $(r \neq 0)$, and x is irrational, prove that $r + x$ and rx are irrational.